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Abstract. This paper studies the asymptotic behavior of a particle with large initial velocity and subject to a force field which is randomly time dependent and inhomogenous in space. We analyze the diffusive limit $\epsilon \to 0$ of the position–velocity pair under the appropriate space-time rescaling: $\big(\epsilon^3 Y(s/\epsilon^2), \epsilon \dot{Y}(s/\epsilon^2)\big)$. Two alternative approaches are proposed. The first one is based on hydrodynamic limits and homogenization techniques for the underlying kinetic equation; the second one on homogenization of the random distribution of trajectories. Time randomness is embodied into an underlying Markov process. Space inhomogeneity is modeled by a periodic structure in the first approach, and by a random field in the second one. In the first case, the analysis relies on the dissipation properties of the Markov process, whereas in the second one, the mixing properties of the random field are used. We point out more analogies and differences of the two obtained results.

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1 Introduction

This paper studies the effective long time behavior of a single particle (or a set of independent particles) in a time random and spatially inhomogenous force field. The initial particle velocity is assumed to be large compared to the typical scales of the force field. This problem is motivated by the general study of transport properties of a particle classically coupled to a specific environment or a thermostat as described in the series of papers [4], [6]–[8]. In the present work, the back reaction of the particle on the environment, responsible for dissipation of the particle energy, is neglected, as in the forthcoming paper [2]. We also expect that the environment alone firstly has some appropriate internal dissipation and fluctuation mechanisms, and secondly has a non vanishing space average. This appears when considering a particle excited by a randomly oscillating force field with some space inhomogeneities. It can model for instance a charged particle excited by an oscillating electric force, and interacting with a crystal that causes the space inhomogeneity. Such a context arises in laser-matter interaction modeling. Similar problems arise when considering a particle subject to a drag force from a surrounding turbulent flow under some appropriate scaling assumptions [20]. The physical intuition in this case

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without back reaction is as follows: the kinetic energy of the particle is activated, and the latter follows a random walk on velocity whose spatial probability distribution has a mass going to infinity at large time. Thus, the main mathematical problem consists in finding an appropriate time-space scaling under which the particle exhibits an effective (i.o.w. homogeneized) diffusive dynamical behavior with respect to its velocity.

In what follows, one is interested in force fields which are inhomogenous with respect to space, and random with respect to time. More precisely, denoting by

$$s \in \mathbb{R}, \quad y \in \mathbb{R}^d, \quad u \in \mathbb{R}^d,$$

the microscopic variables that respectively stand for time, position and particle velocity, the force field is a time and space function

$$(s, y) \mapsto \mathscr{F}_s(y)$$

that satisfies the two following properties:

- At any time s, the space dependence presents an homogenous local average (typically non vanishing), denoted by:
$$\langle \mathscr{F}_s \rangle \quad (\neq 0).$$

- The time dependence $s \mapsto \langle \mathscr{F}_s \rangle$ is modeled by a random mixing process with vanishing average:
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \mathscr{F}_s \rangle \, \mathrm{d}s = 0 \quad a.s.$$

Here and below, the local average $\langle \cdot \rangle$ is:

- either a periodic average (periodic model); the analysis being carried out on the transport PDE level using two-scale convergence techniques,

- or the average over some additional randomness of the field $y \mapsto \mathscr{F}_s(y)$ (disordered random model); the analysis applying to the probability distribution of the particle path, using tightness and martingale characterization techniques.

The random time dependence of the field is restricted to exponentially decreasing correlations in time, so that it is natural to model this time dependence by a Markov process (of dimension 1, for notational simplicity only). The force driving the particles at time s is then given by

$$\mathscr{F}_s = \mathscr{F}(., Q_s),$$

where in the above $s \mapsto Q_s \in \mathbb{R}$ denotes the process at hand, and the force field is described by a two variables function:

$$(y, q) \mapsto \mathscr{F}(y, q).$$

One then supposes appropriate long time mixing properties of the Markov process $s \mapsto Q_s$ with respect to a stationary Maxwellian probability distribution $\mathscr{M}(q)\,\mathrm{d}q$, with $\mathscr{M}$ a normalized positive function. The Markovian evolution of the process will be described by an operator $\mathcal{Q}$, which stands for the usual Markov generator. One will suppose the existence of its adjoint operator $\mathcal{Q}^\star$ defined for the inner product in $L^2(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q)$:

$$(f, g) = \int_{\mathbb{R}} f(q)g(q)\ \mathscr{M}(q)\ \mathrm{d}q.$$

Assume the initial microscopic velocity u_0 is of order $1/\epsilon$, and consider now the macroscopic position/time variables (x, t), defined by the following time-space re-scaling:

$$\begin{cases} x = \epsilon^3 y \\ t = \epsilon^2 s\ , \end{cases} \tag{1}$$

with scaling parameter $\epsilon > 0$. Accordingly, the velocity is rescaled as

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \epsilon u = \epsilon \frac{\mathrm{d}y}{\mathrm{d}s} = \mathcal{O}(1).$$

The rationale behind this scaling comes from the central limit theorem for Markov processes [3] in the homogeneous case ($\mathscr{F}$ does not depend on position). Indeed, given the initial microscopic state (y_0, u_0), the velocity of the particle obeys

$$u(s) - u_0 = \int_0^s \mathscr{F}(Q_{s'})\ \mathrm{d}s' \underset{s\to\infty}{\sim} C\sqrt{s},$$

and thus the position is driven by

$$y(s) - y_0 = \int_0^s u(s')\ \mathrm{d}s' \underset{s\to\infty}{\sim} \frac{2}{3} C s^{3/2}.$$

The scaling (1) we adopt is then such that the rescaled quantities remains of order $\mathcal{O}(1)$ with respect to ϵ on a fixed time interval $0 \le t \le T$. Then the problem under consideration becomes the study of the limiting behavior when $\epsilon \to 0$ of the process

$$t \mapsto (X_t^\epsilon, V_t^\epsilon),$$

solution to the differential equation:

$$\begin{cases} \dfrac{\mathrm{d}X_t^\epsilon}{\mathrm{d}t} = V_t^\epsilon \\ \\ \dfrac{\mathrm{d}V_t^\epsilon}{\mathrm{d}t} = \dfrac{1}{\epsilon}\mathscr{F}\left(\dfrac{X_t^\epsilon}{\epsilon^3}, Q_{t/\epsilon^2}\right). \end{cases} \tag{2}$$

On the basis of the above classical heuristic, or using a more formal perturbative analysis which will be detailed below, one can expect a pure diffusive behavior of the dynamics:

$$t \mapsto V_t^\epsilon,$$

in the limit $\epsilon \to 0$, with a constant diffusion matrix given by:

$$\mathscr{D} := \frac{1}{2}(\tilde{\mathscr{D}} + \tilde{\mathscr{D}}^T),$$

where the coefficients are given by the Kubo formula (similarly to the space homogenous case, see [3, 14] for instance). In this context, the Kubo formula is simply given by the averaged time auto-correlation of the homogeneized force field:

$$\tilde{\mathscr{D}} = -\int_{\mathbb{R}} \mathbb{E}\left(\langle \mathscr{F} \rangle (Q_0) \otimes \langle \mathscr{F} \rangle (Q_t)\right)\ \mathrm{d}t. \tag{3}$$

As a remarkable fact, the diffusion coefficient can be equivalently recast using the Markov generator of the time random dynamics as

$$\tilde{\mathscr{D}} = -\int_{\mathbb{R}} \big(\langle \mathscr{F} \rangle (q) \otimes \mathcal{Q}^{-1}(\langle \mathscr{F} \rangle)(q)\big)\, \mathscr{M}(q)\ \mathrm{d}q. \tag{4}$$

In (2), the ϵ-asymptotics combines the fast oscillations of the force field, both with respect to time and space, with its large amplitude of order $1/\epsilon$. This result has first to be compared to the classical diffusion approximation for random evolutions, which appears in (2) when the field is spatially homogenous: as explained above, it yields a diffusion behavior with a diffusion coefficient given by the time auto-correlation as in (3). Secondly, we can also compare with the case of purely space dependent models which are considered in the classical and so-called "Landau diffusive limit" of the stochastic acceleration problem. Dealing with a force field with a vanishing spatial average, the problem has been studied in [23], and revisited by many authors, see in particular the analysis of the two-dimension case in [12, 24] and the references therein. The regime of the Landau diffusion emerges by considering the space-time scaling

$$\begin{cases} x = \epsilon^4 y \\ t = \epsilon^3 s\ , \end{cases}$$

instead of (1). Note that in the above references one usually starts from a "weak coupling" of order $\mathcal{O}(\epsilon)$ between the particle and the field, yet this choice is strictly equivalent up to re-scaling to the "large initial velocity" of order $\mathcal{O}(1/\epsilon)$ we use here. In the specific case where the force field derives from a potential, the effective dynamical behavior of the momentum of the particle is then a Landau diffusion, that is to say a diffusion on a sphere, which conserves the kinetic energy at the macroscopic scale. However, in this regime, the physical meaning of the asymptotic process differs from the one we wish to investigate since the corresponding diffusion matrix relies on the space correlations of the field. Thus in the analysis of the latter regime, the time dependence of the force field turns out to be unecessary, except for artificial technical purposes (which can simplify the proof as in [30]). The results of the "Landau diffusive limit" have been strengthened in [10], with a sharp non-asymptotic description of the large time behavior of the particle, at least for space dimension larger than 4 and for Poisson fields that do not derive from a potential. Yet, the "Landau diffusion" limit is not the only possible limit theorem for stochastic acceleration models, and in this spirit, the passive transport problem addressed in [22] has been studied in [25] in the slow decorrelation regime leading to a fractional Brownian motion limit. Finally, we mention the "strong field regime" which would correspond in our presentation to the case where the force derives from a potentiel, and the initial velocity is no longer large, but of order $\mathcal{O}(1)$. A possible scaling is then

$$\begin{cases} x = \epsilon y \\ t = \epsilon s\ . \end{cases}$$

This regime is much more difficult to analyze, and does not lead to diffusion effects: the problem has been addressed in [15], and we refer to the breakthrough on this question due to [5].

As said above, this paper deals with the classical diffusion approximation, where the diffusive behavior is driven by the time auto-correlation of a time dependent force, and requires the scaling (1). The novelty comes from the space inhomogeneities, see (3), which has to be resolved at the most refined time scale. This work is also the opportunity to compare the two classical homogeneization approaches. The former focuses on the path-wise stochastic behavior, the latter deals with the underlying transport PDE. Therefore, beyond the obtained convergence statement, our aim is also to present a self-contained parallel treatment of a homogenization problem by using different mathematical toolboxes. One of the difficulty of the analysis can be explained as follows. A naive approach would lead to consider the transport equation

$$v \cdot \nabla_y \phi = \psi$$

where, in practice, the right hand side depends on the force field, and has null spatial average. However, as it is well known, the transport operator is not Fredholm and the inversion usually does not make sense. For this reason, a naive construction of oscillating test functions which would mimic the formal derivation does not work directly. In both the stochastic and the PDE approaches an additional argument is required, based either on the mixing properties of the random field or on strengthened entropy estimates.

The stochastic pathwise homogeneization of the dynamical system (2) is made possible by modeling the inhomogenous force field by a random field, and by looking at the probability distribution of the random path with this additional randomness. The classical analysis, see [23, 24], relies on the fact that in dimension $d \geq 3$, the position path never intersects itself. Then, it is possible to use the mixing property of the random field to define suitably averaged quantities, involving formally the solution of the transport equation. In the present work, we propose a simplified approach by constructing a suitable oscillating test function. Then, the difficulty consists in obtaining the necessary sharp estimates with respect to ϵ. In dimension $d = 2$, it is certainly possible to adapt the proof in the spirit of [24, 12]; in dimension 1, one has to restrict to path with signed momentum ($v > 0$), and what happens when momentum vanishes remains an open question.

The limit theorem at the transport PDE level is based on the kinetic interpretation of the model which, by contrast to the probabilistic approach, is directly tractable using classical double-scale convergence arguments. To this end, let us introduce[1]

$$f^\epsilon(t, x, v, q) \geq 0$$

[1] It could be convenient to change the unknown by setting

$$g(t, x, v, q) = f(t, x, v, q)\ \mathscr{M}(q)$$

which is now a density distribution with respect to the standard Lebesgue measure $\mathrm{d}v\ \mathrm{d}x\ \mathrm{d}q$. It still satisfies (5) but replacing the right hand side by

$$\mathscr{M}(q)\ \mathcal{Q}^\star(g).$$

The proof can be adapted to this framework, at the price of using weighted spaces for the operators $\mathcal{Q}/\mathcal{Q}^\star$.

the density with respect to $\mathrm{d}x\ \mathrm{d}v\ \mathscr{M}(q)\ \mathrm{d}q$ in $\mathbb{R}^{2d} \times \mathbb{R}$ of the probability distribution of the random variable $(X_t^\epsilon, V_t^\epsilon, Q_{t/\epsilon^2})$. It means that, given measurable sets $\mathscr{X} \subset \mathbb{R}^d$, $\mathscr{V} \subset \mathbb{R}^d$ and $\mathscr{K} \subset \mathbb{R}$, we have, at time $t \geq 0$,

$$\int_{\mathscr{X}\times\mathscr{V}\times\mathscr{K}} f^\epsilon(t,x,v,q)\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x = \mathrm{Proba}\Big(\big\{(X_t^\epsilon, V_t^\epsilon, Q_{t/\epsilon^2}) \in \mathscr{X}\times\mathscr{V}\times\mathscr{K}\big\}\Big).$$

Accordingly, the differential system (2) translates into the following evolution PDE on densities:

$$\partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon + \frac{1}{\epsilon}\mathscr{F}(x/\epsilon^3, q)\cdot\nabla_v f^\epsilon = \frac{1}{\epsilon^2}\mathscr{Q}^\star(f^\epsilon). \tag{5}$$

The equation is completed by imposing the Cauchy data

$$f^\epsilon(t=0,x,v,q) = f^\epsilon_{\mathrm{Init}}(x,v,q), \tag{6}$$

where $f^\epsilon_{\mathrm{Init}}(x,v,q)$ is a non negative integrable function given by the initial probability density distribution of $(X^\epsilon_{\mathrm{Init}}, V^\epsilon_{\mathrm{Init}}, Q_{\mathrm{Init}})$. The problem under consideration becomes the behavior as $\epsilon \to 0$ of the solutions $f^\epsilon(t,x,v,q) \geq 0$ of (5). Equation (5) gives another way of considering the scaling (1), rather in the spirit of an asymptotic regime. Start from the PDE

$$\partial_s f + u\cdot\nabla_y f + F(y,q)\cdot\nabla_u f = \frac{1}{\tau}\mathscr{Q}^\star(f) \tag{7}$$

which is equivalent to the trajectory description, written with dimension variables. Then, we introduce "macroscopic" observation time and length scales T and L, respectively, which are the observation units. It defines the velocity scale L/T. The random force field is characterized by :

- the relaxation time τ associated to the Markov process $s \mapsto Q_s$,
- a typical length scale ℓ of the variation of the field,
- the amplitude F_0 of the force.

We write (7) in dimensionless form, and we obtain (5) under the following assumptions:

$$\tau = \epsilon^2\mathrm{T}, \qquad \ell = \epsilon^3\mathrm{L}$$

while the amplitude of the force is $F_0 = \frac{1}{\epsilon}\ \frac{\mathrm{L}}{\mathrm{T}^2}$. If one thinks ℓ and τ as microscopic length and time scales, this assumption means that the microscopic velocity ℓ/τ scales like ϵ. It is also natural to relate the amplitude of the force to the microscopic units, saying $F_0 = \ell/\tau^2$, which is indeed consistent with the scaling assumptions. This makes the bridge with the relation (1). Moreover, this interpretation is also related to a possible motivation for studying such an asymptotic problem. It arises when considering a particle interacting with a crystal and excited by an external source: ℓ appears as the length scale of the lattice and $1/\tau$ stands for the frequency of the excitation.

In contrast to the standard kinetic framework, we are dealing with an extended phase space, according to (2), where the additional variable q can be interpreted as a "state" variable for the particle. Transport acts on the space-velocity variables (x,v) while the relaxation effect acts here on the variable q only. The stiffest terms in (5) will impose a specific behavior of the unknown with respect to the variable q, in the spirit of the so–called "hydrodynamic regimes", see e. g. [16]. This aspect is reminiscient to the modeling adopted in [11] for describing turbulence "seen from the particles" in dispersed two–phase flows through the introduction of an additional hidden variable which naturally leads to hydrodynamic type regimes. The analysis is then quite natural since it mimics the formal development that allows to guess the limit. It is based on classical tools from homogenization theory (double-scale convergence [1, 29], oscillating test functions...) and hydrodynamic limits, in the spirit of [18, 19]. We shall see however that some technical restriction appear with this purely deterministic method.

The paper is organized as follows. In Section 2 we make more precise the framework in which we perform the analysis. In particular, we will detail the necessary assumptions on the generator $\mathcal{Q}$ and on the force field $\mathscr{F}$, pointing out the differences between the PDE and the probabilistic approach. In Section 3, we guess on formal grounds the asymptotic behavior as ϵ goes to 0. Then, we state precisely the results we obtain. The proofs are detailed in Section 4 and 5. An appendix of independent interest details a possible relevant extension of the functional framework.

2 Notations and Assumptions

2.1 General setting

Recall that $\mathcal{Q}$ is a Markov operator and satisfies the mass conservation identity:

$$\mathcal{Q}(1) = 0 \qquad \text{or} \qquad \int_{\mathbb{R}} \mathcal{Q}^\star(f)\mathscr{M}(q)\ \mathrm{d}q = 0. \tag{8}$$

In the same way, the Maxwellian probability distribution $\mathscr{M}(q)\ \mathrm{d}q$ (where, throughout the paper $\mathscr{M}$ is a positive function) is a stationary distribution with respect to $\mathcal{Q}$:

$$\int_{\mathbb{R}} \mathcal{Q}(f)\mathscr{M}(q)\ \mathrm{d}q = 0 \qquad \text{or} \qquad \mathcal{Q}^\star(1) = 0. \tag{9}$$

Classical mixing/dissipation properties of $\mathcal{Q}$ or $\mathcal{Q}^\star$ will be assumed, which traduces exponentially fast relaxation to the equilibrium distribution $\mathscr{M}(q)\ \mathrm{d}q$, that is to say

$$\mathrm{e}^{t\mathcal{Q}}(\cdot) \underset{\exp}{\overset{t\to+\infty}{\longrightarrow}} \int_{\mathbb{R}} \cdot\, \mathscr{M}(q)\ \mathrm{d}q$$

in an appropriate functional space (say L^2 or L^∞). It is worth pointing out that the set of assumptions we need will be satisfied when dealing with the following operators which are relevant on both the mathematical and physical viewpoints:

$$\text{the Fokker-Planck operator} \qquad \mathcal{Q}(f) = \mathscr{M}^{-1}\partial_q\big(\mathscr{M}\partial_q(f)\big), \tag{10}$$

$$\text{the linear Boltzmann operator} \qquad \mathcal{Q}(f) = \int_{\mathbb{R}} f(q')\mathscr{M}(q')\ \mathrm{d}q' - f(q), \tag{11}$$

In these examples, one has usually $\mathscr{M}(q) = \frac{1}{\sqrt{2\pi\theta}}\, e^{-q^2/2\theta}$. The operator (10) is associated to the diffusion $s \mapsto Q_s$ solution to the stochastic differential equation:

$$\mathrm{d}Q_s = \nabla \ln \mathscr{M}(q)\ \mathrm{d}s + \sqrt{2}\ \mathrm{d}W_s,$$

where $(W_s)_{s\geq 0}$ is a usual Brownian motion, while (11) is associated to a jump process $s \mapsto Q_s$ verifying:

$$\begin{cases} (\tilde{Q}_n)_{n\geq 1} \text{ are i.i.d. random variables with distribution } \mathscr{M}(q')\ \mathrm{d}q', \\ (T_n - T_{n-1})_{n\geq 1} \text{ are i.i.d. random variables with exponential distribution,} \\ Q_t = Q_0 \text{ for } t \in [T_0 = 0, T_1[, \\ Q_t = \tilde{Q}_n \text{ for } t \in [T_n, T_{n+1}[. \end{cases}$$

2.2 Assumptions, PDE approach

In this Section, we detail the assumptions needed when working in the PDE framework. The mixing requirement on the Markov operators $\mathcal{Q}^\star/\mathcal{Q}$ is usual, and will consist in a spectral gap assumption which states as follows

$$\begin{cases} \text{There exists } \sigma > 0 \text{ such that} \\ -\displaystyle\int_{\mathbb{R}} \mathcal{Q}(f)\ f\mathscr{M}(q)\ \mathrm{d}q \geq \sigma \int_{\mathbb{R}} \left| f(q) - \int_{\mathbb{R}} f(q')\mathscr{M}(q')\ \mathrm{d}q' \right|^2 \mathscr{M}(q)\ \mathrm{d}q \geq 0. \end{cases} \tag{12}$$

It makes $L^2(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q)$ — which clearly embeds into $L^1(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q)$ — a natural functional space for investigating the spectral properties of $\mathcal{Q}$. Note that (12) has the following useful consequence:

$$\mathrm{Ker}(\mathcal{Q}) \text{ (resp. } \mathrm{Ker}(\mathcal{Q}^\star)) \text{ has dimension one and is spanned by constant functions.}$$

To carry out the analysis of (5) in a L^2 setting, one only needs a weak regularity assumption on $\mathcal{Q}$, namely the Fredholm alternative:

$$\begin{cases} \text{For any } h \in L^2(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q) \text{ verifying } \int_{\mathbb{R}} h\mathscr{M}(q)\ \mathrm{d}q = 0, \\ \text{there exists a unique solution } g \in L^2(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q) \text{ of} \\ \mathcal{Q}(g) = h \text{ (resp. } (\mathcal{Q}^\star(g) = h) \text{ and } \int_{\mathbb{R}} g\mathscr{M}(q)\ \mathrm{d}q = 0. \end{cases} \tag{13}$$

In most reasonable cases, (13) is a consequence of the spectral gap inequality (12); for instance when $\mathcal{Q}$ is a bounded operator on $L^2(\mathbb{R}, \mathscr{M}(q)\ \mathrm{d}q)$, (13) follows from a direct application of the Lax-Milgram Theorem. Anyway, (13) holds for the operators (10) and (11) we have in mind throughout the paper.

To set up the hypothesis on the force field $\mathscr{F}$, one needs a few notations: in what follows $\mathbb{Y}$ stands for the unit cube $(0,1)^d$ and the symbol # means that we consider $\mathbb{Y}-$ periodic functions. We suppose

$$y \longmapsto \mathscr{F}(y,q) \quad \text{is } \mathbb{Y}\text{-periodic,} \tag{14}$$

$$\sup_{y\in\mathbb{Y},\ q\in\mathbb{R}} |\mathscr{F}(y,q)| \leq C < \infty, \tag{15}$$

$$\text{for a.e } y \in \mathbb{Y}, \quad \int_{\mathbb{R}} \mathscr{F}(y,q)\ \mathscr{M}(q)\ \mathrm{d}q = 0. \tag{16}$$

Note that the centering assumption (16) is quite restrictive. It will be relaxed in the stochastic framework.

2.3 Assumptions, stochastic approach

As said in the Introduction, the stochastic approach relies on considering a random force field, through the dependence with respect to an additional "environment" variable. The stochastic homogenization resorting crucially to the mixing properties of the field. To be more specific, we introduce a new probability space modeling the field randomness

$$(\Omega_e, \mathscr{F}_e, P_e) \quad \text{a probability space,}$$

and we denote

$$\langle \phi \rangle = \int_{\Omega_e} \phi(\omega_e)\ \mathrm{d}P_e(\omega_e),$$

the average with respect to the realisations $\omega_e \in \Omega_e$. If $(\Omega_m, \mathscr{F}_m, P_m)$ denotes the probability space on which the Markov process $t \mapsto Q_t$ with generator $\mathcal{Q}$ is defined, then the full probability space is simply the product of the two:

$$(\Omega, \mathscr{F}, \mathbb{P}) := (\Omega_e, \mathscr{F}_e, P_e) \times (\Omega_m, \mathscr{F}_m, P_m).$$

Then, we consider a force field depending on the three variables (space, Markov process and random environment)

$$\begin{array}{rlcl} \mathscr{F}: & \mathbb{R}^d \times \mathbb{R} \times \Omega_e & \longrightarrow & \mathbb{R}^d \\ & (y, q, \omega_e) & \longmapsto & \mathscr{F}(y, q, \omega_e). \end{array}$$

When considering the field $\mathscr{F}$ as a random variable, the dependance with respect to ω_e will be omitted. The key qualitative property of the field we assume is homogeneity on average:

$$\text{For any } q \in \mathbb{R}, \text{ the average } \langle \mathscr{F} \rangle(y, q) \text{ is space homogeneous,} \tag{17}$$

which means it does not depend on the variable y:

$$\langle \mathscr{F} \rangle(y, q) = \langle \mathscr{F} \rangle(q).$$

We also still assume centering with repect to time dependence:

$$\int_{\mathbb{R}} \langle \mathscr{F} \rangle(q) \mathscr{M}(q)\ \mathrm{d}q = 0. \tag{18}$$

Next, for any given Borel subset $\Lambda \subset \mathbb{R}^d$, we denote

$$\langle \cdot | \Lambda \rangle$$

the conditional averaging with respect to the sigma-field (intuitively the "information") $\mathscr{G}_\Lambda$ generated by the random field $\mathscr{F}(y, q, \omega_e)$ for $(y, q) \in \Lambda \times \mathbb{R}$. In other words, $\mathscr{G}_\Lambda$ stands for the minimal σ-algebra included in $\mathscr{F}_e$ generated by sets of the form $\{\omega_e \in \Omega,\ \mathscr{F}(y, q, \omega_e) \in A\}$, for A ranging in Borel sets of $\mathbb{R}^d$, $y \in \Lambda$, and $q \in \mathbb{R}$. Then, it is useful to evaluate the decorrelation of the force field between two distinct sets. To this end, we set

$$\alpha(\Lambda_1, \Lambda_2) := \sup_{|Z| \leq 1} |\langle Z | \Lambda_1 \rangle - \langle Z \rangle|,$$

for Z $\mathscr{G}_{\Lambda_2}$-measurable. The decorrelation function (see also [23, 24]) is then:

$$\beta(r) = \sup_{d(\Lambda_1,\Lambda_2)\geq r} \alpha(\Lambda_1, \Lambda_2),$$

d being the usual distance between two sets. Denoting by

$$\|\mathscr{F}\|_\infty (\omega_e) := \sup_{y,q} |\mathscr{F}(y, q, \omega_e)|,$$

we assume almost sure smoothness and boundedness of the force field and its gradient:

$$\begin{cases} (y, q) \mapsto \mathscr{F}(y, q, \omega_e) \text{ and } (y, q) \mapsto \nabla_y \mathscr{F}(y, q, \omega_e) \\ \text{are } \omega_e\text{-a.s. continuous and } \|\mathscr{F}\|_\infty \text{ and } \|\nabla_y \mathscr{F}\|_\infty \text{ have finite variance.} \end{cases} \tag{19}$$

One has then the following decorrelation estimates which will be needed:

Lemma 2.1 *Let $\Lambda \subset \mathbb{R}^d$. One has for any $y, q \in \mathbb{R}^d \times \mathbb{R}$:*

$$|\langle(\mathscr{F} - \langle\mathscr{F}\rangle)(y, q)|\Lambda\rangle| \leq 2 \|\mathscr{F}\|_\infty \beta(d(y, \Lambda)) \quad P_e - a.s.,$$

as well as

$$\|\langle\nabla_y \mathscr{F}(y, q)|\Lambda\rangle\| \leq \|\nabla_y \mathscr{F}\|_\infty \beta(d(y, \Lambda)) \quad P_e - a.s..$$

Proof. The result follows by remarking that $(\mathscr{F} - \langle\mathscr{F}\rangle)(y, q)$ and $\nabla_y \mathscr{F}(y, q)$ have null average and are $\mathscr{G}_{V_y}$-measurable, V_y being any small neigbourhood of y. Then the result follows from the continuity of fields and the definition of the decorrelation rate. $\square$

Finally we (only) require quasi-linear decorrelation rate:

$$r\beta(r) \xrightarrow[r\to\infty]{} 0. \tag{20}$$

A typical example of a random field verifying the above hypothesis is given by the following description. Let f_{int} be a smooth field on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ and set:

$$\mathscr{F}(y, q, \omega_e) := \sum_{n\in\mathbb{Z}^d} f_{\text{int}}(y - n, q, \omega_e^n),$$

where $\omega_e := (\omega_e^n)_{n\in\mathbb{Z}^d}$, P_e being a product measure (independance of sites), and the average $\langle f_{\text{int}}\rangle(y, q) = \langle f_{\text{int}}\rangle(q)$ is independent of y and centered in q. Then the decorrelation assumption ((20)) is trivially satisfied when f_{int} is compactly supported; and can be achieved with sufficient polynomial decrease at infiinity.

It is likely that the boundedness assumption on the field (19), which is also used in [24] for instance, may be relaxed, yet probably at the price of higher technicalities. Yet it is worth emphasizing that the technical cost, and the quasi-linear de-correlation assumption (20) are substantially less demanding compared to classical references [23, 24], for the Landau diffusion limit case. Eventually, note that the centering assumption (16) on the force field used in the PDE approach is no longer necessary, since it has been replaced by random inhomogeneity.

We can now make precise the assumption on the generator $\mathcal{Q}$. The functional framework slightly differs from the previous section and for the auxiliary equation we require

$$\left\{\begin{array}{l}\text{There is a solution } \chi \text{ of the cell problem:}\\ \mathcal{Q}\chi = -\langle \mathscr{F} \rangle\\ \text{belonging to } L^{\infty}(\mathbb{R}) \cap C^{0}(\mathbb{R}).\end{array}\right. \tag{21}$$

Hypothesis (21) is immediately satisfied for instance when geometric uniform convergence occurs:

$$\left\| \mathrm{e}^{t\mathcal{Q}}(\phi) - \int_{\mathbb{R}} \phi(q)\mathscr{M}(q)\ \mathrm{d}q \right\|_{\infty} \leq C\mathrm{e}^{-\kappa t}\left\|\phi\right\|_{\infty},$$

for some $C, \kappa > 0$. Conditions on $\mathcal{Q}$ under which the latter occurs have been thoroughly studied (see e.g. [28] and references therein), and includes examples (10) and (11), at least when $-\ln \mathscr{M}(q) = \mathcal{O}_{|q|\to+\infty}(q^2)$.

Dimension 1 The case of dimension 1 is treated in a different and rather elementary way by inverting the transport operator. Neither periodicity nor randomness of the force field need to be assumed. The single required assumption reads as follows:

$$\left\{\begin{array}{l}\text{There is a solution } \lambda : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{*} \to \mathbb{R} \text{ to the Poisson equation:}\\ v\partial_y \lambda(y,q,v) = \mathscr{F}(y,q) - \langle \mathscr{F} \rangle\\ \text{with } \lambda \text{ and } \partial_v\lambda \text{ being continuous and bounded on each closed definition subset.}\end{array}\right. \tag{22}$$

3 Formal Analysis and Main Results

We guess the asymptotic behavior by inserting in (5) the following double-scale Hilbert expansion

$$f^{\epsilon}(t,x,v,q) = \sum_{j\geq 0} \epsilon^{j} F^{(j)}(t,x,x/\epsilon^3,v,q)$$

where the functions $F^{(j)}$ are supposed $\mathbb{Y}-$periodic[2] with respect to the third variable. Using the expansion modifies the advection term according to $v\cdot\nabla_x \to v\cdot\nabla_x + \frac{1}{\epsilon^3}v\cdot\nabla_y$. Then, we identify terms with the same power of ϵ and we get

$$\epsilon^{-3} \text{ terms:} \qquad v\cdot\nabla_y F^{(0)} = 0, \tag{23}$$

$$\epsilon^{-2} \text{ terms:} \qquad v\cdot\nabla_y F^{(1)} = \mathcal{Q}^{\star}(F^{(0)}), \tag{24}$$

$$\epsilon^{-1} \text{ terms:} \qquad v\cdot\nabla_y F^{(2)} = \mathcal{Q}^{\star}(F^{(1)}) - \mathscr{F}(y,q)\cdot\nabla_v F^{(0)}, \tag{25}$$

$$\epsilon^{0} \text{ terms:} \qquad v\cdot\nabla_y F^{(3)} = \mathcal{Q}^{\star}(F^{(2)}) - \big(\partial_t F^{(0)} + v\cdot\nabla_x F^{(0)} + \mathscr{F}(y,q)\cdot\nabla_v F^{(1)}\big). \tag{26}$$

For the time being, we analyze the cell equations only at the formal level, neglecting completely any possible technical difficulty related to the solvability of these cell problems. With this in mind, we infer from (23) that the leading term $F^{(0)}$ does not depend on the fast variable y. Therefore, due to the periodic boundary condition, integrating (24) with respect to y yields

$$\mathcal{Q}^{\star}(F^{(0)}) = 0$$

[2]At this formal level, the periodicity assumption could be generalized to homogeneity at a large scale.

which implies, due to (9), that $F^{(0)} = \rho^{(0)}(t, x, v)$ where $\rho^{(0)}$ does not depend on q. In turn, (24) becomes $v \cdot \nabla_y F^{(1)} = 0$ and the first order corrector does not depend on the fast variable anymore. Then, integration of (25) with respect to y leads to

$$\mathscr{Q}^\star(F^{(1)}) = \int_{\mathbb{Y}} \mathscr{F}(y, q)\ \mathrm{d}y \cdot \nabla_v F^{(0)} = \int_{\mathbb{Y}} \mathscr{F}(y, q)\ \mathrm{d}y\ \ \cdot \nabla_v \rho^{(0)}.$$

Owing to (13) and (16), we can find a vector valued function $\chi^\star = (\chi_1^\star, \dots, \chi_d^\star) \in \left[L^2(\mathbb{R}; \mathscr{M}\ \mathrm{d}q)\right]^d$ verifying

$$\mathscr{Q}^\star(\chi_k^\star)(q) = -\int_{\mathbb{Y}} \left[\mathscr{F}(y, q)\right]_k \mathrm{d}y\ , \qquad \int_{\mathbb{R}} \chi^\star(q)\mathscr{M}(q)\ \mathrm{d}q = 0. \tag{27}$$

For further purposes, we also introduce $\chi(q)$ the solution of the adjoint equation

$$\mathscr{Q}(\chi)(q) = -\int_{\mathbb{Y}} \mathscr{F}(y, q)\ \mathrm{d}y, \qquad \int_{\mathbb{R}} \chi(q)\mathscr{M}(q)\ \mathrm{d}q = 0. \tag{28}$$

Note that (12) implies

$$\int_{\mathbb{R}} |\chi(q)|^2 \mathscr{M}(q)\ \mathrm{d}q \le \frac{1}{\sigma^2}\|\mathscr{F}\|^2_{L^\infty(\mathbb{Y}\times\mathbb{R})}, \qquad \int_{\mathbb{R}} |\chi^\star(q)|^2 \mathscr{M}(q)\ \mathrm{d}q \le \frac{1}{\sigma^2}\|\mathscr{F}\|^2_{L^\infty(\mathbb{Y}\times\mathbb{R})}.$$

This auxiliary function yields the following expression for the corrector

$$F^{(1)}(t, x, v, q) = -\chi^\star(q) \cdot \nabla_v \rho^{(0)}(t, x, v). \tag{29}$$

We deduce the evolution equation satisfied by $\rho^{(0)}$ by integrating (26) with respect to both y and q; we get

$$\begin{aligned} \partial_t \rho^{(0)} + v \cdot \nabla_x \rho^{(0)} &= -\nabla_v \cdot \left(\int_{\mathbb{R}} \int_{\mathbb{Y}} \mathscr{F}(y, q) F^{(1)}\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q\right) \\ &= \nabla_v \cdot \left(\int_{\mathbb{R}} \int_{\mathbb{Y}} \mathscr{F}(y, q) \otimes \chi^\star(q)\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q\ \nabla_v \rho^{(0)}\right), \end{aligned}$$

namely we are led to a Fokker-Planck equation with the effective diffusion matrix

$$\tilde{\mathscr{D}} = \int_{\mathbb{R}} \left(\int_{\mathbb{Y}} \mathscr{F}(y, q)\ \mathrm{d}y\right) \otimes \chi^\star(q)\mathscr{M}(q)\ \mathrm{d}q = -\int_{\mathbb{R}} \mathscr{Q}^\star(\chi^\star) \otimes \chi^\star\ \mathscr{M}(q)\ \mathrm{d}q. \tag{30}$$

This enables to define the symmetric part of the above matrix:

$$\mathscr{D} := \frac{1}{2}(\tilde{\mathscr{D}}^T + \tilde{\mathscr{D}}).$$

Note that only the symmetric part $\tilde{\mathscr{D}}^T + \tilde{\mathscr{D}}$ is involved in the limiting Fokker-Planck equation. We check that the coefficient is indeed non negative.

Lemma 3.1 *The matrix $\tilde{\mathscr{D}}$ verifies, for any $\xi \in \mathbb{R}^d$, $\tilde{\mathscr{D}}\xi \cdot \xi \ge 0$. Besides, we remark that*

$$\tilde{\mathscr{D}} = -\int_{\mathbb{R}} \chi \otimes \mathscr{Q}(\chi)\ \mathscr{M}(q)\ \mathrm{d}q.$$

If $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y = 0$*, the matrix* $\tilde{\mathscr{D}}$ *is actually* 0*, otherwise, there exists* $\xi \in \mathbb{R}^d$ *verifying* $\tilde{\mathscr{D}}\xi \cdot \xi > 0$*. If furthermore the matrix*

$$\mathscr{A} = \int_{\mathbb{R}} \Big(\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y \Big) \otimes \Big(\int_{\mathbb{Y}} \mathscr{F}(y',q)\ \mathrm{d}y' \Big)\ \mathscr{M}(q)\ \mathrm{d}q$$

is invertible, then there exists $\delta > 0$ *such that for any* $\xi \in \mathbb{R}^d$ *we have* $\tilde{\mathscr{D}}\xi \cdot \xi \geq \delta|\xi|^2$*.*

Proof. Let $\xi \in \mathbb{R}^d \setminus \{0\}$. We rewrite

$$\tilde{\mathscr{D}}\xi \cdot \xi = -\int_{\mathbb{R}} \mathcal{Q}^\star(\chi^\star \cdot \xi)\ \chi^\star \cdot \xi\ \mathscr{M}\ \mathrm{d}q$$

which is non-negative by (12). It vanishes iff $\chi^\star(q) \cdot \xi = \Lambda$, $\Lambda \in \mathbb{R}$. But then, we have $\int_{\mathbb{R}} \chi^\star(q) \cdot \xi \mathscr{M}(q)\ \mathrm{d}q = 0 = \Lambda$. It implies that $\chi^\star(q) \cdot \xi = 0$ and coming back to (27), we get $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y \cdot \xi = 0$. Therefore, $\tilde{\mathscr{D}}\xi \cdot \xi$ is positive for any direction ξ which is not orthogonal to $\big\{ \int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y,\ q \in \mathbb{R} \big\}$. Eventually, we remark that

$$\mathscr{A}\xi \cdot \xi = \int_{\mathbb{R}} \Big(\int_{\mathbb{Y}} \mathscr{F}(y,q) \cdot \xi\ \mathrm{d}y \Big)^2 \mathscr{M}(q)\ \mathrm{d}q$$

is positive for any $\xi \in \mathbb{R}^d \setminus \{0\}$ when $\mathscr{A}$ is invertible which forces $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y \cdot \xi \neq 0$ for a.e. $q \in \mathbb{R}$. Accordingly, we get $\tilde{\mathscr{D}}\xi \cdot \xi > 0$. □

The main result in the L^2 framework of the paper states as follows.

Theorem 3.2 *Assume that* (8)–(9) *and* (12)–(16) *are fulfilled. We suppose that the initial condition verifies*

$$\sup_{\epsilon > 0} \int_{\mathbb{R}^d \times \mathbb{R}^d \times R} |f^\epsilon_{\mathrm{Init}}(x,v,q)|^2 \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x \leq C < \infty. \tag{31}$$

Then, up to a subsequence, f^ϵ *solutions of* (5) *associated to* $f^\epsilon_{\mathrm{Init}}$ *converges weakly in* $L^2((0,T) \times \mathbb{R}^d \times \mathbb{R}^d \times R; \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x)$ *to* $\rho(t,x,v)$*, where* ρ *is the solution of*

$$\left\{ \begin{array}{l} \partial_t \rho + v \cdot \nabla_x \rho = \nabla_v \cdot (\mathscr{D} \nabla_v \rho), \\ \rho(t=0,x,v) = \text{weak}-\lim\limits_{\epsilon \to 0} \displaystyle\int_{\mathbb{R}} f^\epsilon_{\mathrm{Init}}(x,v,q) \mathscr{M}(q)\ \mathrm{d}q, \end{array} \right. \tag{32}$$

with $\mathscr{D}$ *defined by* (30)*.*

The main result of the paper in the probabilistic framework states as follows.

Theorem 3.3 *Let* $d \geq 3$*, and suppose* (17)–(21)*. Consider the stochastic process defined on the full probability space* $(\Omega, \mathscr{F}, \mathbb{P})$*:*

$$t \mapsto (X^\epsilon_t, V^\epsilon_t),$$

solution of (2) *for* $t \in [0,T]$*. This induces a probability distribution* $\mathcal{P}^\epsilon$ *on the space of continuous trajectories* $C^0([0,T], \mathbb{R}^{2d})$ *endowed with uniform convergence. Suppose the initial* *state* $(X^\epsilon_{\text{Init}}, V^\epsilon_{\text{Init}})$ *converges in law towards a given probability distribution* $\mu_{\text{Init}}(\,\mathrm{d}v\ \mathrm{d}x)$. *Assume that* $\mu_{\text{Init}}(v=0) = \mu_{\text{Init}}(v^T \mathscr{D} v = 0) = 0$. *Then* $\mathcal{P}^\epsilon$ *converges in distribution on* $C^0([0,T], \mathbb{R}^{2d})$ *towards the probability distribution* $\mathcal{P}$ *which is defined as follows: Trajectories*

$$t \mapsto (X_t, V_t)$$

are initially distributed according to μ_{Init}, $\frac{\mathrm{d}X_t}{\mathrm{d}t} = V_t$, *and* $t \mapsto V_t$ *is a Wiener process with diffusion matrix* $\mathscr{D}$.

In dimension 1, we get

Theorem 3.4 *Let* $d = 1$, *and suppose (21)-(22). Consider the stochastic process for* $t \in [0,T]$:

$$t \mapsto (X_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon}),$$

solution of (2), and stopped at time:

$$\tau^\epsilon = \inf\{t \in [0,T] | V^\epsilon_t = 0\}.$$

This defines a probability distribution $\mathcal{P}^\epsilon$ *on the space of continuous trajectories* $C^0([0,T], \mathbb{R}^2)$ *endowed with uniform convergence. Suppose the initial state* $(X^\epsilon_{\text{Init}}, V^\epsilon_{\text{Init}})$ *converges in law towards a given probability distribution* $\mu_{\text{Init}}(\mathrm{d}v\ \mathrm{d}x)$ *such that* $\mu_{\text{Init}}(v=0) = 0$. *Then* $\mathcal{P}^\epsilon$ *converges in distribution on* $C^0([0,T], \mathbb{R}^2)$ *towards the probability distribution* $\mathcal{P}$ *which is defined as follows: Trajectories*

$$t \mapsto (X_t^\tau, V_t^\tau)$$

stopped at $\tau = \inf\{t \in [0,T] | V_t = 0\}$ *are initially distributed according to* μ_{Init}, *and obey* $\frac{\mathrm{d}X_t}{\mathrm{d}t} = V_t$, *with* $t \mapsto V_t$ *a Wiener process with diffusion constant* $\mathscr{D}$.

Remark 3.5 *These statements should be completed by a couple of remarks:*

- *In both cases, it is crucial to prevent from null initial macroscopic velocity. This corresponds in original microscopic variables to particles with large initial velocity* $(\dot{y}(t=0) \sim 1/\epsilon)$.

- *The main difference between Theorem 3.2 and Theorem 3.3 comes from the modeling of the force field, which is periodic in the former case, and random in the latter. The probablistic approach of Theorem 3.3 and 3.4 does not need the restrictive centering condition (16). The price to pay is that Theorem 3.3 is a result* on average *with respect to field randomness* ω_e. *Almost sure convergence may hold but remains an open question.*

- *For technical reasons, which are explained below, the statement in Theorem 3.2 excludes the situation where* $f^\epsilon_{\text{Init}}(x,v,q)$ *is (or converges to) a Dirac mass with respect to the space and velocity variables. Nevertheless, it is possible to adapt the functional framework in order to deal with* L^1 *functions, the point being to exclude concentration phenomena. We give some hints in that direction in the Appendix. In Theorem 3.3 the convergence* $\mathrm{Law}(X^\epsilon_t, V^\epsilon_t) \to \mathrm{Law}(X_t, V_t)$ *allows to consider Dirac distributed random variables, including initial conditions.*

- *In the proof of Theorem 3.2, we shall prove that* $\int f^\epsilon(t,x,v,q)\mathscr{M}(q)\ \mathrm{d}q$ *converges in* $C^0([0,T];L^2(\mathbb{R}^d\times\mathbb{R}^d)-\text{weak})$ *that is*

  $$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\mathbb{R}} f^\epsilon(t,x,v,q)\mathscr{M}(q)\ \mathrm{d}q\right)\ \varphi(x,v)\ \mathrm{d}v\ \mathrm{d}x = \int_{\mathbb{R}^d\times\mathbb{R}^d}\rho(t,x,v)\ \varphi(x,v)\ \mathrm{d}v\ \mathrm{d}x$$

  holds for any $\varphi\in L^2(\mathbb{R}^d\times\mathbb{R}^d)$ *uniformly on* $[0,T]$, *as soon as the initial condition in* (32) *indeed makes sense. This can be compared to Theorem 3.3 as follows: the force field is now random, and the density* f^ϵ *has to be integrated with respect to the environment randomness* ω_e. *More precisely Theorem 3.3 implies as a corollary that uniformly with respect to* $t\in[0,T]$, *the probability distribution of* $(X_t^\epsilon, V_t^\epsilon)$ *converges weakly towards the probability distribution of* (X_t,V_t) *given (when the density exists) by* $\rho(t,x,v)$. *In other words,*

  $$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^d}&\left(\int_{\mathbb{R}\times\Omega_e} f^\epsilon(t,x,v,q,\omega_e)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}P_e(\omega_e)\right)\varphi(x,v)\ \mathrm{d}x\ \mathrm{d}v\\ &=\int_{\mathbb{R}^d\times\mathbb{R}^d}\rho(t,x,v)\ \varphi(x,v)\ \mathrm{d}v\ \mathrm{d}x\end{aligned}$$

  holds for any φ *continuous and bounded, uniformly on* $[0,T]$, *as soon as the corresponding initial condition indeed makes sense.*

- *Theorem 3.4 is specific to dimension* 1. *There, the force field can be determinisitic, at the price of considering paths for positive momentum* $v>0$ *only. The distinction in Theorem 3.3 and 3.4, between dimension* $d=1$, $d=2$, *and* $d\geq 3$, *comes from the possibility of self-intersections of the position path* $t\mapsto X_t^\epsilon$. *It is certainly possible to extend Theorem 3.3 for dimension* $d=2$, *with the same assumption, by preventing tangential self-intersection, in the spirit of [24].*

- *Note that we do not exclude the case where* $\mathscr{D}=0$, *which holds for instance when the force field derives from a potential since it implies* $\int_{\mathbb{Y}}\mathscr{F}(y,q)\ \mathrm{d}y=0$. *In such a case* (32) *becomes a mere transport equation and we actually have* $\rho(t,x,v)=\rho^{\mathrm{Init}}(x-tv,v)$. *Yet under the assumptions of Theorem 3.2, when the matrix* $\mathscr{A}$ *in Lemma 3.1 is invertible, the limit verifies* $\nabla_v\rho\in L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$.

- *It is possible to consider force fields depending on both the fast (that is* x/ϵ^3*) and the slow (that is* x*) variables; we skip the tedious details for such an adaptation.*

4 Proof of Theorem 3.2

The proof starts by obtaining uniform estimates.

Proposition 4.1 *Let the assumptions of Theorem 3.2 be fulfilled. Then, the sequences* f^ϵ *satisfy*

$$\sup_{\epsilon>0}\left(\sup_{t\geq 0}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}|f^\epsilon|^2\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\right)\leq C<\infty,$$

and we also have

$$0\leq\sup_{\epsilon>0}\left(-\frac{1}{\epsilon^2}\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}\mathscr{Q}^\star(f^\epsilon)\ f^\epsilon\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\right)\leq C<\infty.$$

Corollary 4.2 *We can write the ansatz*

$$\begin{cases} f^\epsilon(t,x,v,q) = \rho^\epsilon(t,x,v) + \epsilon r^\epsilon(t,x,v,q), \\ \rho^\epsilon(t,x,v) = \displaystyle\int_{\mathbb{R}} f^\epsilon(t,x,v,q)\ \mathscr{M}(q)\ \mathrm{d}q, \qquad \int_{\mathbb{R}} r^\epsilon(t,x,v,q)\ \mathscr{M}(q)\ \mathrm{d}q = 0, \end{cases}$$

where the remainder r^ϵ is bounded in $L^2((0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R};\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t)$.

Proof. The proof combines the specific differential structure of the left hand side of (5) and the dissipation property of the operator $\mathcal{Q}$. On the one hand, we have

$$\Big(v\cdot\nabla_x f^\epsilon + \frac{1}{\epsilon}\mathscr{F}(x/\epsilon^3,q)\cdot\nabla_v f^\epsilon\Big) f^\epsilon \ = \frac{1}{2}\left(\nabla_x\cdot\big(v\ |f^\epsilon|^2\big) + \frac{1}{\epsilon}\nabla_v\cdot\big(\mathscr{F}(x/\epsilon^3,q)|f^\epsilon|^2\big)\right)$$

the integral over $\mathbb{R}^d\times\mathbb{R}^d$ of which vanishes. On the other hand, (12) leads to

$$\int_{\mathbb{R}} \mathcal{Q}^\star(f^\epsilon)\ f^\epsilon\ \mathscr{M}\ \mathrm{d}q \le 0.$$

Putting the pieces together yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} |f^\epsilon|^2\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x - \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} \mathcal{Q}^\star(f^\epsilon)\ f^\epsilon\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x = 0,$$

which justifies the statement by virtue of (12). □

To make the formal derivation devised above rigorous, we shall use the framework of double–scale convergence as introduced in [1, 29] (see adaptations to vector-valued functions in [18]): with Proposition 4.1, possibly at the price of extracting subsequences (but we still denote the considered subsequence with the index ϵ, with a slight abuse of notation) we can suppose that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} & f_\epsilon(t,x,v,q)\ \varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}} F(t,x,y,v,q)\ \varphi(t,x,y,v,q)\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\end{aligned}$$

holds for any smooth enough trial function, say $\varphi\in C^0_{c,\#}([0,\infty)\times\mathbb{R}^d\times\mathbb{Y}\times\mathbb{R}^d;L^2(\mathbb{R};\mathscr{M}(q)\ \mathrm{d}q))$, where the symbol # means that we assume periodicity with respect to the third variable. Similarly, by Corollary 4.2, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} & r_\epsilon(t,x,v,q)\ \varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}} R(t,x,y,v,q)\ \varphi(t,x,y,v,q)\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

The double scale limits F and R belong to $L^2_\#((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{Y}\times\mathbb{R};\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}y\ \mathrm{d}x\ \mathrm{d}t)$ for any $0<T<\infty$. Furthermore, since $f^\epsilon-\rho^\epsilon=\epsilon r^\epsilon$ converges strongly to 0, we check that

$$\begin{cases} f^\epsilon(t,x,v,q) \rightharpoonup \displaystyle\int_{\mathbb{Y}} F(t,x,y,v,q)\ \mathrm{d}y = \rho(t,x,v)\ \mathbb{1}(q) \\ \text{weakly in } L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R};\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t) \end{cases}$$

where
$$\rho(t,x,v) = \int_{\mathbb{R}} \int_{\mathbb{Y}} F(t,x,y,v,q)\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d \times \mathbb{R}^d))$$
also coincides with the weak limit of $\rho^\epsilon(t,x,v)$.

Since we have already understood that f^ϵ essentially behaves like its "hydrodynamical part" $\rho^\epsilon(t,x,v)$, we average the equation over the variable q; we get the following "moment equation"

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{R}} f^\epsilon \mathscr{M}(q)\ \mathrm{d}q\Big) + \nabla_x \cdot \Big(\int_{\mathbb{R}} v f^\epsilon \mathscr{M}(q)\ \mathrm{d}q\Big) + \nabla_v \cdot \Big(\int_{\mathbb{R}} \frac{1}{\epsilon}\mathscr{F}(x/\epsilon^3, q)\ f^\epsilon \mathscr{M}(q)\ \mathrm{d}q\Big) = 0\\ = \partial_t \rho^\epsilon + \nabla_x \cdot (v\rho^\epsilon) + \nabla_v \cdot \Big(\int_{\mathbb{R}} \mathscr{F}(x/\epsilon^3, q)\ r^\epsilon\ \mathscr{M}(q)\ \mathrm{d}q\Big),\end{aligned} \tag{33}$$

where, owing to (16), we have remarked that
$$\int_{\mathbb{R}} \frac{1}{\epsilon}\mathscr{F}(x/\epsilon^3,q)\ f^\epsilon \mathscr{M}(q)\ \mathrm{d}q = \int_{\mathbb{R}} \mathscr{F}(x/\epsilon^3,q)\ \frac{f^\epsilon - \rho^\epsilon}{\epsilon}\ \mathscr{M}(q)\ \mathrm{d}q = \int_{\mathbb{R}} \mathscr{F}(x/\epsilon^3,q) r^\epsilon\ \mathscr{M}(q)\ \mathrm{d}q.$$
Letting $\epsilon \to 0$ in (33) yields
$$\partial_t \rho + \nabla_x \cdot (v\rho) + \nabla_v \cdot \Big(\int_{\mathbb{R}} \int_{\mathbb{Y}} \mathscr{F}(y,q)\ R(t,x,y,v,q)\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}y\Big) = 0, \tag{34}$$
and we are thus left with the task of identifying the double scale limit R of the fluctuation r^ϵ. Before, let us remark the following important compactness result.

Lemma 4.3 *The sequence ρ^ϵ is relatively compact in $C^0([0,T]; L^2(\mathbb{R}^d \times \mathbb{R}^d) - weak)$.*

Proof. This property follows from formula (33) together with the estimates in Proposition 4.1. Indeed, for any given function $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ we check readily that $\int_{\mathbb{R}^d \times \mathbb{R}^d} \rho^\epsilon \varphi\ \mathrm{d}v\ \mathrm{d}x$ is equibounded and equicontinuous. Hence, by virtue the Arzela-Ascoli Theorem it lies in a compact set of $C^0([0,T])$. An approximation argument allows to extend this property to any given function $\varphi \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. We conclude by using a standard diagonal reasoning. □

The method consists in multiplying the equation (5) by suitable oscillating trial functions in the spirit of [17, 19]. At first, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ \varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ \partial_t\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &\quad + \frac{1}{\epsilon^3}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ v\cdot\nabla_y\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ v\cdot\nabla_x\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &\quad + \frac{1}{\epsilon}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ \mathscr{F}(x/\epsilon^3,q)\cdot\nabla_v\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &\quad + \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} f_\epsilon(t,x,v,q)\ \mathcal{Q}(\varphi)(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x.\end{aligned} \tag{35}$$

Therefore, as ϵ goes to 0 we obtain

$$\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_{\mathbb{R}} \int_{\mathbb{Y}} F(t,x,y,v,q)\ v\cdot\nabla_y\varphi(t,x,y,v,q)\ \mathrm{d}y\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = 0,$$

from which we deduce $v\cdot\nabla_y F = 0$. Accordingly the Fourier coefficients verify $v\cdot k\ \widehat{F}(t,x,k,v,q) = 0$. Since for any $k\in\mathbb{Z}^d\setminus\{0\}$ and a.e $v\in\mathbb{R}^d$, $v\cdot k\neq 0$, we conclude that F does not depend on y (see Remark 4.4 below). Thus, from now on we write

$$F = F(t,x,v,q) = \rho(t,x,v)\ \mathbb{1}(q),$$

which fits with the first step of the formal analysis.

Next, we remark that the second and the fifth term in the right hand side of (35) can be recast as

$$\frac{1}{\epsilon^3}\int_{\mathbb{R}^d\times\mathbb{R}^d} \rho_\epsilon(t,x,v)\ v\cdot\nabla_y\Big(\int_{\mathbb{R}} \varphi\mathscr{M}(q)\ \mathrm{d}q\Big)(t,x,x/\epsilon^3,v)\ \mathrm{d}v\ \mathrm{d}x \\ +\frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} r_\epsilon(t,x,v,q)\ v\cdot\nabla_y\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x$$

and

$$\frac{1}{\epsilon}\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} r_\epsilon(t,x,v,q)\ \mathscr{Q}(\varphi)(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x$$

respectively. Let us pick a trial function verifying the constraint

$$\int_{\mathbb{R}} \varphi\mathscr{M}\ \mathrm{d}q = 0. \tag{36}$$

For such a function, multiplying (35) by ϵ^2 we are led to

$$\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} r_\epsilon(t,x,v,q)\ v\cdot\nabla_y\varphi(t,x,x/\epsilon^3,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \\ \xrightarrow[\epsilon\to 0]{} \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}} R(t,x,y,v,q)\ v\cdot\nabla_y\varphi(t,x,y,v,q)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}t = 0$$

However, $\int_{\mathbb{R}} r^\epsilon\mathscr{M}\ \mathrm{d}q = 0$ implies that $\int_{\mathbb{R}} R\mathscr{M}\ \mathrm{d}q = 0$ too and we deduce that, for any test function (not necessarily verifying (36)),

$$\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}} R\ v\cdot\nabla_y\varphi\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}t \\ = \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}} R\ v\cdot\nabla_y\Big(\varphi(q) - \int_{\mathbb{R}}\varphi(q')\mathscr{M}(q')\ \mathrm{d}q'\Big)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}t = 0$$

holds. Accordingly R does not depend on the fast variable y anymore.

Then, we consider a test function $\varphi(t,x,v,q)$ which does not depend on the fast variable. Multiplying (35) by ϵ we get

$$\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\Big(r_\epsilon\ \mathscr{Q}\varphi + f_\epsilon\ \mathscr{F}(x/\epsilon^3,q)\cdot\nabla_v\varphi\Big)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \\ \xrightarrow[\epsilon\to 0]{} \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\int_{\mathbb{Y}}\Big(R\ \mathscr{Q}\varphi + \rho\ \mathscr{F}(y,q)\cdot\nabla_v\varphi\Big)\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}t = 0 \\ = \int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}}\Big[R\ \mathscr{Q}\varphi + \rho\ \Big(\int_{\mathbb{Y}}\mathscr{F}(y,q)\ \mathrm{d}y\Big)\cdot\nabla_v\varphi\Big]\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = 0. \tag{37}$$

The interest of this relation is two fold: first it induces some regularity information on ρ, second it yields the necessary expression of R by means of ρ.

If $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y = 0$, then (37) actually tells us that R belongs to $\mathrm{Ker}(Q)$ and thus $R(t,x,v,q) = \widetilde{\rho}(t,x,v)\ \mathbb{1}(q)$. Then, due to (16), the last term in (34) vanishes and ρ satisfies a mere free transport equation.

Let us now assume that $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y \neq 0$. We make use of $\chi(q)$ defined by (28). We set $\varphi(t,x,v,q) = \psi(t,x,v)\ \chi(q)$, with $\psi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. We observe that (37) leads to

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_{\mathbb{R}} \rho\Big(\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y\Big)\cdot\nabla_v\varphi\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\\
&\qquad = -\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \rho\Big(\int_{\mathbb{R}} \mathscr{Q}\chi\otimes\chi\mathscr{M}(q)\ \mathrm{d}q\Big)\nabla_v\psi\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\\
&\qquad = -\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \rho\ \tilde{\mathscr{D}}^T\nabla_v\psi\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = -\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}} R\ \mathscr{Q}\varphi\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\\
&\qquad = -\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\mathbb{R}} R\ \mathscr{Q}\chi\mathscr{M}(q)\ \mathrm{d}q\Big)\psi\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\\
&\qquad = -\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\int_{\mathbb{R}}\int_{\mathbb{Y}} R\ \mathscr{F}(y,q)\mathscr{M}(q)\ \mathrm{d}q\Big)\psi\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}
$$

It follows that

$$
\Big|\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \rho\ \tilde{\mathscr{D}}^T\nabla_v\psi\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\Big| \leq C\|\mathscr{F}\|_{L^\infty(\mathbb{Y}\times\mathbb{R})}\ \|\psi\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}.
$$

Hence, when $\tilde{\mathscr{D}}$ is invertible, we conclude that $\nabla_v\rho$ lies in $L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$, identifying a regularizing effect induced by the asymptotics. Anyway, the regularizing effect holds in the directions where $\tilde{\mathscr{D}}$ is not degenerate (see Lemma 3.1).

Then, we go back to (37) considering a trial function which separates variables $\varphi(t,x,v,q) = \phi(q)\psi(t,x,v)$. Since we can write $\int_{\mathbb{Y}} \mathscr{F}(y,q)\ \mathrm{d}y = -\mathscr{Q}^\star(\chi^\star)$, (37) becomes

$$
\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_{\mathbb{R}} \Big(R\psi - \rho\chi^\star\cdot\nabla_v\psi\Big)\ \mathscr{Q}(\phi)\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = 0.
$$

Let $\eta \in L^2(\mathbb{R};\mathscr{M}\ \mathrm{d}q)$ verifying $\int_{\mathbb{R}} \eta\ \mathrm{d}q = 0$. By virtue of (13) it can be rewritten as $\mathscr{Q}(\phi) = \eta$ so that

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_{\mathbb{R}} \Big(R\psi - \rho\chi^\star\cdot\nabla_v\psi\Big)\ \eta\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = 0\\
&= \int_{\mathbb{R}} \Big[\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(R\psi - \rho\chi^\star\cdot\nabla_v\psi\Big)\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\Big]\ \eta\ \mathscr{M}(q)\ \mathrm{d}q
\end{aligned}
$$

holds for any such η. But, since by definition

$$
\int_{\mathbb{R}} R\mathscr{M}\ \mathrm{d}q = 0 = \int_{\mathbb{R}} \chi^\star\mathscr{M}\ \mathrm{d}q
$$

this relation actually extends to any $\eta \in L^2(\mathbb{R};\mathscr{M}\ \mathrm{d}q)$. We deduce that

$$
\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(R\psi - \rho\chi^\star\cdot\nabla_v\psi\Big)\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t = 0 \qquad \text{a.e. } q\in\mathbb{R}.
$$

In particular it follows that

$$\left| \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho \chi^\star \cdot \nabla_v \psi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \right| \leq \|\psi\|_{L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)} \; \|R(\cdot, q)\|_{L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)}$$

which implies

$$\mathrm{div}_v(\chi^\star(q)\rho(t,x,v)) = \chi^\star(q) \cdot \nabla_v \rho(t,x,v) \in L^2((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}; \mathscr{M} \, \mathrm{d}q \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t).$$

It finally proves

$$R(t,x,v,q) = -\chi^\star(q) \cdot \nabla_v \rho(t,x,v).$$

Plugging this formula into (33) yields the expected diffusion equation. $\square$

Remark 4.4 *In the latter proof, we should care of the functional framework: when F is a function (i.e. it is absolutely continuous with respect to the Lebesgue measure) we deduce from $v \cdot \nabla_y F = 0$ that F does not depend on y. This conclusion does not apply when F is only supposed to be a bounded measure on $\mathbb{R}^d \times \mathbb{Y}$: for any periodic function $g(y)$, the distribution $F(y,v) = g(y)\delta_{v=0}$ satisfies $v \cdot \nabla_y F = 0$.*

5 Proof of Theorem 3.3 and Theorem 3.4

The proof is very classical, and uses cut-off/tightness/martingale arguments, in this usual order (see the classical monographs [14, 21], as well as references therein). In the case where the force field $\mathscr{F}$ does not depend on space, the present problem is solved by the classical diffusion approximation results that can be found in the forementioned references. Here, the introduction of randomness in space is very similar to the stochastic acceleration problem in the classical paper [23]. Yet, in the present work, appropriate compensating test functions in the spirit of [21] are introduced, which considerably simplify the technical handling of the asymptotic analysis.

5.1 General setting

Probabilistic proofs are carried out by considering sequences of probability distributions over a functional space of trajectories, in the present context:

$$\mathscr{C}_T := C^0([0,T], \mathbb{R}^d, \sup_{t\in[0,T]} \|\cdot\|).$$

Sequences of random processes indexed by ϵ, like the velocity trajectory $t \mapsto V_t^\epsilon$, induces sequences of probability distributions over $\mathscr{C}_T$ that may converge[3]. The relative compactness (or tightness) of ϵ-sequences of probability distributions on $\mathscr{C}_T$ is usually proven with a random version of the Arzela-Ascoli compactness criteria, for instance the Kurtz-Aldous criterion.

Lemma 5.1 (Kurtz-Aldous) *ϵ-sequences of probability distributions on $\mathscr{C}_T$ induced by a random process $t \mapsto V_t^\epsilon \in \mathbb{R}^d$ are tight if the following two conditions are satisfied:*

[3]Convergence of probability measures is weak over bounded and continuous test functions

1. *ϵ-sequences of the probability distribution on $\mathbb{R}$ induced by the random variable $\sup_{t\in[0,T]}|V_t^\epsilon|$ are tight.*

2. *For any smooth compactly supported test function ϕ, and any $\delta > 0$ there is a random modulus of continuity γ_δ^ϵ such that for any $t \in [0, T]$:*

$$\sup_{h\in[0,\delta]} \mathbb{E}\left(\left(\phi(V_{t+h}^\epsilon) - \phi(V_t^\epsilon)\right)^2 |(V_s^\epsilon)_{s\in[0,t]}\right) \leq \mathbb{E}\left(\gamma_\delta^\epsilon|(V_s^\epsilon)_{s\in[0,t]}\right) \quad a.s.,$$

where $\mathbb{E}\left(\,.\,|(V_s^\epsilon)_{s\in[0,t]}\right)$ denotes the conditional expectation given the path $(V_s^\epsilon)_{s\in[0,t]}$ and γ_δ^ϵ verifies

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\mathbb{E}(\gamma_\delta^\epsilon) = 0.$$

Proof. See [14] Chapter 3: Tightness of ϵ-sequences of $t \mapsto V_t^\epsilon \in R^d$ in the Skorohod space is given by Theorem 8.6. Tightness of the full process is given by Theorem 9.1. One gets tightness on $\mathscr{C}_T$ using for instance Therorem 10.2 and Problem 25. $\square$

It is very helpful to remember that if such a convergence occurs, by the Skohorod embedding theorem, it is *equivalent* (and very useful) to construct (in the probabilistic jargon) an "abstract underlying probability space", and to consider the whole of an ϵ-sequence of random processes $t \mapsto V_t^\epsilon$ converging (uniformly in time) *almost surely* (i.o.w. with probability 1) towards a random continuous process $t \mapsto V_t$. Since in our case the position is given by $X_t^\epsilon = X_0^\epsilon + \int_0^t V_{t'}^\epsilon \, \mathrm{d}t'$, the convergence of ϵ-sequences of the process $t \mapsto V_t^\epsilon$ induces convergence for pair $t \mapsto (X_t^\epsilon, V_t^\epsilon)$.

Now, the proof of convergence relies on four steps:

1. First step: Contain the random evolution (using a stopping time τ^ϵ, and a cut-off parameter $\eta > 0$) in a clever domain where the formal analysis presented in Section 3 can be made rigorous. The resulting stopped process will be denoted by:

$$t \mapsto (X_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon}) := (X_{t\wedge\tau^\epsilon}^\epsilon, V_{t\wedge\tau^\epsilon}).$$

2. Second step: Show relative compactness of the probability distribution induced by the stopped process $t \mapsto V_t^{\epsilon,\tau^\epsilon}$. To this end, we shall use a compensating (or perturbed) test function method, see [21]. Here, random environments are considered, represented by the realisations $\omega_e \in \Omega_e$ associated with the probability space of the environment $(\Omega_e, \mathcal{F}_e, P_e)$. The compensating test function argument needs to be adapted through the following Lemma.

   Lemma 5.2 *For a given realisation ω_e of the environment, suppose that the full stopped process $t \mapsto (X_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon}) \in \mathbb{R}^{2d}$ is a continuous Markov process with a generator $\mathcal{L}^\epsilon$ depending on a multi-dimensional random field $\mathscr{F} := \mathscr{F}(.,\omega_e)$. At each smooth and compactly supported test function $\phi \in C_0^\infty(\mathbb{R}^d)$, one associates a "compensating pertubation" given by a random (depending on ω_e and measurable) continuous function $\phi^\epsilon(\cdot,\omega_e) \in C^0(\mathbb{R}^{2d})$. Suppose the following conditions are satisfied:*

   (a) The stopped process $t \mapsto V_t^{\epsilon,\tau^\epsilon}$ remains in a compact set.

(b) For any test function $\phi \in C_0^\infty(\mathbb{R}^d)$, there is a deterministic constant $\gamma_\phi^\epsilon > 0$ with $\lim_{\epsilon\to 0} \gamma_\phi^\epsilon = 0$, such that for any $t \in [0,T]$,

$$\left|\mathbb{E}\left(\phi(V_t^{\epsilon,\tau^\epsilon}) - \phi^\epsilon(X_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon})|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right)\right| \leq \gamma_\phi^\epsilon \quad a.s..$$

(c) For any test function $\phi \in C_0^\infty(\mathbb{R}^d)$, there is a $\gamma_\phi > 0$, such that for any $t \in [0,T]$,

$$\left|\mathbb{E}\left(\mathcal{L}^\epsilon(\phi^\epsilon)(X_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon})|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right)\right| \leq \gamma_\phi \quad a.s..$$

Then ϵ-sequences of the stopped process $t \mapsto V_t^{\epsilon,\tau^\epsilon}$ are tight in $\mathscr{C}_T$.

Proof. Denote $\phi_t := \phi(V_t^{\epsilon,\tau^\epsilon})$ and $\phi_t^\epsilon := \phi^\epsilon(V_t^{\epsilon,\tau^\epsilon}, X_t^{\epsilon,\tau^\epsilon})$; $\phi^{2,\epsilon}$ denotes the compensating perturbed test function obtained from ϕ^2. The key consists in applying Lemma 5.1, by introducing the compensating perturbed test functions, for $0 \leq t \leq t+h \leq T$:

$$\begin{aligned}
&(\phi_{t+h} - \phi_t)^2 \\
&\quad = \phi_{t+h}^{2,\epsilon} - \phi_t^{2,\epsilon} + 2\phi_t(\phi_{t+h}^\epsilon - \phi_t^\epsilon) \\
&\qquad + R_{t+h}.
\end{aligned}$$

with

$$R_{t+h} \quad = \quad (\phi_{t+h}^2 - \phi_{t+h}^{2,\epsilon}) - (\phi_t^2 - \phi_t^{2,\epsilon}) + 2\phi_t(\phi_{t+h} - \phi_{t+h}^\epsilon - \phi_t + \phi_t^\epsilon).$$

Now by assumption, one gets the estimate:

$$\left|\mathbb{E}\left(R_{t+h}|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right)\right| \leq 2\gamma_{\phi^2}^\epsilon + 2\left\|\phi\right\|_\infty \gamma_\phi^\epsilon.$$

In the same way, using the martingale property of Markov processes for each ω_e and integrating over the latter, one has:

$$\begin{aligned}
&\mathbb{E}\left(\phi_{t+h}^\epsilon - \phi_t^\epsilon|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right) \\
&\quad = \mathbb{E}\left(\int_t^{t+h} 1_{\tau^\epsilon\geq t'}\mathcal{L}^\epsilon(\phi^\epsilon)(X_{t'}^{\epsilon,\tau^\epsilon}, V_{t'}^{\epsilon,\tau^\epsilon})dt'|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right).
\end{aligned}$$

Since the process $t \mapsto V_t^\epsilon$ is continuous and $t \mapsto X_t^\epsilon$ is a deterministic function of the former, the event $\{\tau^\epsilon \geq t'\}$ is measurable with respect to the path $(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t']}$, and thus:

$$\begin{aligned}
&\mathbb{E}\left(\phi_{t+h}^\epsilon - \phi_t^\epsilon|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right) \\
&\quad = \int_t^{t+h} \mathbb{E}\left(1_{\tau^\epsilon\geq t'}\mathbb{E}\left(\mathcal{L}^\epsilon(\phi^\epsilon)(X_{t'}^{\epsilon,\tau^\epsilon}, V_{t'}^{\epsilon,\tau^\epsilon})|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t']}\right)|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right) dt'.
\end{aligned}$$

Finally, by assumption one gets the estimate:

$$\left|\mathbb{E}\left((\phi_{t+h} - \phi_t)^2|(V_s^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right)\right| \leq h\gamma_{\phi^2} + 2\left\|\phi\right\|_\infty h\gamma_\phi + 2\gamma_{\phi^2}^\epsilon + 2\left\|\phi\right\|_\infty \gamma_\phi^\epsilon,$$

and the Kurtz-Aldous criteria in Lemma 5.1 applies. □

3. Third step: One extracts a converging sub-sequence, and identify the limit by using the so-called characterizing "martingale" problem. The "martingale" problem is simply a characterizing set of conserved quantities (on average) by a Markov random evolution. This can be seen as a random generalisation, with a dual expression, of the characteristic equations for transport PDE's.

Definition 5.3 *The continuous random process $t \mapsto V_t$ is said to verify the stopped martingale problem with Markov generator $\mathcal{L}$ and stopping time τ, if for any time ladder $t_0 = 0 < t_1 < ... < t_n < t_{n+1} < T$, any smooth test functional with compact support Φ, and any smooth test function with compact support ϕ:*

$$\mathbb{E}\left(\Phi(V^{\tau}_{t_0}, ..., V^{\tau}_{t_n})(\phi(V^{\tau}_{t_{n+1}}) - \phi(V^{\tau}_{t_n}) - \int_{t_n}^{t_{n+1}} \mathcal{L}(\phi)(V_t) 1_{\tau \geq t} \, \mathrm{d}t)\right) = 0.$$

If $\mathcal{L} = \operatorname{div}(\mathcal{D}\nabla \cdot)$, the latter characterizes Brownian motions with diffusion coefficient $\mathcal{D}$, and stopped at τ (see Theorem 6.1 Chapter 4 in [14]).

Proving the martingale property for the limit of a sequence constructed from Lemma 5.2 can be done by using the following:

Lemma 5.4 *Consider the context of Lemma 5.2, extract a converging sub-sequence and assume the following:*

(a) Convergence in probability distribution of the pair

$$((V^{\epsilon,\tau^\epsilon})_{t\in[0,T]}, \tau^\epsilon) \overset{Law}{\rightarrow} ((V^\tau)_{t\in[0,T]}, \tau),$$

for some stopping time τ of $(X^\tau_t, V^\tau_t)_{t\in[0,T]}$.

(b) For any test functions $\phi \in C^\infty_0(\mathbb{R}^d)$, there is a deterministic constant $\gamma^\epsilon_\phi > 0$ with $\lim_{\epsilon\to 0} \gamma^\epsilon_\phi = 0$, such that for any $t \in [0, T]$,

$$\left|\mathbb{E}\left(\mathcal{L}^\epsilon(\phi^\epsilon)(X^{\epsilon,\tau^\epsilon}_t, V^{\epsilon,\tau^\epsilon}_t) - \mathcal{L}(\phi)(V^{\epsilon,\tau^\epsilon}_t) | (V^{\epsilon,\tau^\epsilon})_{s\in[0,t]}\right)\right| \leq \gamma^\epsilon_\phi \quad a.s..$$

Then $t \mapsto V_t$ verifies the stopped martingale problem with Markov generator $\mathcal{L}$ and stopping time τ.

Proof. This proof is using the notations of the proof of Lemma 5.2. Similarly to Lemma 5.2 consider the martingale property for the compensating perturbed test function:

$$\mathbb{E}\left(\Phi(V^{\epsilon,\tau^\epsilon}_{t_0}, ..., V^{\epsilon,\tau^\epsilon}_{t_n})(\phi^\epsilon_{t_{n+1}} - \phi^\epsilon_{t_n} - \int_{t_n}^{t_{n+1}} \mathcal{L}^\epsilon(\phi^\epsilon)(V^{\epsilon,\tau^\epsilon}_t, Z^{\epsilon,\tau^\epsilon}_t) 1_{\tau^\epsilon \geq t} dt)\right) = 0.$$

Then consider a Skorohod explicit representation for which the ϵ-sequence $((V^{\epsilon,\tau^\epsilon})_{t\in[0,T]}, \tau^\epsilon)$ converges almost surely. The martingale property for the limiting process is then obtained using dominated convergence and the assumed estimates. □

4. Fourth step: Remove the localization of step 1 using a posteriori analysis of the limiting process $t \mapsto V_{t \wedge \tau}$. This can be done using the simple following claim.

   Lemma 5.5 *Consider an ϵ-sequence of continuous random processes $(V^\epsilon_t)_{t\in[0,T]}$, as well as its supposed limit $(V_t)_{t\in[0,T]}$. Let η be a cut-off parameter involved in the definition of stopping times τ^ϵ and τ associated with the forementioned processes. Assume the probability distribution of the pair $((V^{\epsilon,\tau^\epsilon})_{t\in[0,T]}, \tau^\epsilon)$ converges towards $((V^\tau)_{t\in[0,T]}, \tau)$ and that:*

   $$\lim_{\eta\to 0} \mathbb{P}(\tau = T) = 1.$$

   Then the probability distribution of the full process $((V^\epsilon)_{t\in[0,T]})$ converges towards $(V_{t\in[0,T]})$.

   Proof. Consider a Skorohod explicit representation for which:

   $$\lim_{\epsilon\to 0} \tau^\epsilon = \tau \quad a.s..$$

   Consider a small $\delta > 0$. By assumption and dominated convergence, for any small $\delta > 0$ there is η_0 and ϵ_{η_0} such that for any $\epsilon < \epsilon_{\eta_0}$:

   $$\mathbb{P}(\tau^\epsilon = T) > 1 - \delta.$$

   Now consider any continuous and bounded functional Φ on $C^0([0,T], \mathbb{R}^d)$ and remark that on the event $\{\tau^\epsilon = T\}$, $\Phi(V^\epsilon_{\cdot\wedge\tau^\epsilon}) = \Phi(V^\epsilon_\cdot)$. Using the assumed convergence $\mathbb{E}\left(\Phi(V^\epsilon_{\cdot\wedge\tau^\epsilon})\right) \to \mathbb{E}\left(\Phi(V_{\cdot\wedge\tau})\right)$, one finally gets for ϵ sufficiently small:

   $$\left|\mathbb{E}\left(\Phi(V^\epsilon_\cdot) - \mathbb{E}\left(\Phi(V_\cdot)\right)\right| < 2\left\|\Phi\right\|_\infty \delta.$$

   □

5.2 Proof of Theorem 3.3

Recall that the random evolution is defined by the full Markov process

$$t \mapsto (Y^\epsilon_t, V^\epsilon_t, Q_{t/\epsilon^2}),$$

whose generator is given by:

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon^3} v \cdot \nabla_y + \frac{1}{\epsilon^2}\mathcal{Q} + \frac{1}{\epsilon}\mathscr{F}(y,q) \cdot \nabla_v.$$

The expected (y,q)-homogeneized limit involves the differential operator:

$$\mathcal{L} = \mathrm{div}_v(\mathcal{D}\nabla_v \cdot).$$

The main ingredients of the proof are the construction of the compensating test function, and the discussion of the estimates that allows to apply the machinery described above.

The compensating perturbed test function Let $\chi(q) = (\chi_1(q), ..., \chi_d(q)) \in \left(C_0\right)^d$ be defined in (21). We set

$$\lambda^\epsilon(v, y, q) = -\int_0^{\theta^\epsilon} \mathrm{e}^{\epsilon t \mathcal{Q}}(\mathscr{F} - \langle \mathscr{F} \rangle)(y + vt, q)\ \mathrm{d}t,$$

where $\lim_{\epsilon\to 0} \theta^\epsilon = +\infty$. Eventually , $\theta^\epsilon \sim 1/\epsilon$ will be taken. By construction, λ^ϵ is solution to:

$$(v \cdot \nabla_y + \epsilon \mathcal{Q})\lambda^\epsilon = (-\mathscr{F} + \langle \mathscr{F} \rangle)(y, q) - \mathrm{e}^{\epsilon\theta^\epsilon \mathcal{Q}}(-\mathscr{F} + \langle \mathscr{F} \rangle)(y + \theta^\epsilon v, q).$$

Then, for a given smooth test function $\phi \in C_0^\infty(\mathbb{R}^d)$, we consider the associated random perturbed test function:

$$\phi^\epsilon(y, v, q) = \phi(v) + \epsilon\chi(q) \cdot \nabla_v \phi(v) + \epsilon^2 \lambda^\epsilon(v, y, q) \cdot \left(\nabla_v \phi + \epsilon \nabla_v (\chi(q) \cdot \nabla_v \phi(v))\right).$$

Indeed

$$\mathcal{L}^\epsilon(\phi + \epsilon\chi \cdot \nabla_v \phi) = \mathscr{F} \cdot \nabla_v(\chi \cdot \nabla_v \phi) + \frac{1}{\epsilon}(\mathscr{F} - \langle \mathscr{F} \rangle) \cdot \nabla_v \phi,$$

which gives out

$$\begin{aligned}
\mathcal{L}^\epsilon(\phi^\epsilon) - \mathcal{L}(\phi) \quad = \quad & \frac{1}{\epsilon}\mathrm{e}^{\epsilon\theta^\epsilon \mathcal{Q}}(\mathscr{F} - \langle \mathscr{F} \rangle)(y + \theta^\epsilon v, q) \cdot \nabla_v \phi \\
& + \mathrm{e}^{\epsilon\theta^\epsilon \mathcal{Q}}(\mathscr{F} - \langle \mathscr{F} \rangle)(y + \theta^\epsilon v, q) \cdot \nabla_v(\chi(q) \cdot \nabla_v \phi) \\
& + \epsilon \mathscr{F} \cdot \nabla_v(\lambda^\epsilon \cdot \nabla_v \phi) + \epsilon^2 \mathscr{F} \cdot \nabla_v(\lambda^\epsilon \cdot \nabla_v(\chi(q) \cdot \nabla_v \phi)). \qquad (38)
\end{aligned}$$

The choice of ϕ^ϵ is motivated by the following remark: by taking formally $\mathrm{e}^{\epsilon\theta\mathcal{Q}}(\mathscr{F} - \langle \mathscr{F} \rangle)(y + \theta v, q) \to 0$ sufficiently fast when $\theta \to +\infty$, one has $\lambda^\epsilon = \mathcal{O}(1)$ and then $\mathcal{L}^\epsilon(\phi^\epsilon) - \mathcal{L}(\phi) = \mathcal{O}(\epsilon)$.

Now the point is to use the mixing properties of the field in order to control the different terms in the above expression.

Key estimates First, boundedness of the force field (19) gives

$$\limsup_{\epsilon\to 0} \frac{\|\lambda^\epsilon\|_\infty}{\theta^\epsilon} < +\infty \quad P_e - a.s.,$$

Assuming that $\theta^\epsilon \ll 1/\epsilon^2$ and using assumption (21) ($\|\chi\|_\infty < +\infty$), we get

$$\|\phi^\epsilon - \phi\|_\infty \underset{\epsilon\to 0}{\to} 0 \quad P_e - a.s.. \qquad (39)$$

Now we want to estimate (38). Denote by $B_{\epsilon,y,v}$ the closed ball centered in $y + \theta_\epsilon v$ and of radius $\theta_\epsilon |v|$. Using the decorrelation property of the force field in Lemma 2.1, one has for $s \in [0, \theta^\epsilon]$ and $|v| \geq \eta_2$,

$$\left|\left\langle \mathscr{F}(y, q) \otimes (\mathscr{F} - \langle \mathscr{F} \rangle)(y + sv, q) | \mathbb{R}^d \setminus B_{\epsilon,y,v} \right\rangle\right| \leq 2\,\|\mathscr{F}\|_\infty^2\, \beta(s\eta_2) \quad P_e - a.s.$$

Since $\mathrm{e}^{\epsilon t \mathcal{Q}}$ is a conservative and positive integral operator, we get

$$\left|\mathrm{e}^{\epsilon t \mathcal{Q}}(\mathscr{F} - \langle \mathscr{F} \rangle)\right| \leq 2\,\|\mathscr{F}\|_\infty \quad P_e - a.s.$$

and using the decorrelation property of the field

$$\left|\left\langle \mathscr{F}(y,q)\otimes\lambda^\epsilon(y,v,q)|\mathbb{R}^d\setminus B_{\epsilon,y,v}\right\rangle\right|\leq 2\,\|\mathscr{F}\|_\infty^2\int_0^{\theta^\epsilon}\beta(s\eta_2)\;\mathrm{d}s\quad P_e-a.s..$$

In the same way, we obtain

$$\left|\left\langle \mathscr{F}(y,q)\otimes\nabla_y\mathscr{F}(y+sv,q)|\mathbb{R}^d\setminus B_{\epsilon,y,v}\right\rangle\right|\leq 2\,\|\mathscr{F}\|_\infty\,\|\nabla_y\mathscr{F}\|_\infty\,\beta(s\eta_2)\quad P_e-a.s.,$$

so that assuming that moreover $\frac{1}{\eta_1}\geq|v|$ it yields

$$\left|\left\langle \mathscr{F}(y,q)\otimes\nabla_v\lambda^\epsilon(y,v,q)|\mathbb{R}^d\setminus B_{\epsilon,y,v}\right\rangle\right|\leq \frac{2}{\eta_1}\,\|\mathscr{F}\|_\infty\,\|\nabla_y\mathscr{F}\|_\infty\int_0^{\theta^\epsilon}\beta(s\eta_2)s\;\mathrm{d}s\quad P_e-a.s..$$

and finally we get for some constant C depending on $\|\phi\|_\infty+\|\nabla\phi\|_\infty$, $\|\chi\|_\infty$, and η_1:

$$\begin{aligned}&\sup_{|v|\in[\eta_2,1/\eta_1],y,q}\left|\langle\mathcal{L}^\epsilon(\phi^\epsilon)(v,y,q)-\mathcal{L}(\phi)(v)|\mathbb{R}^d\setminus B_{\epsilon,y,v}\rangle\right|\\&\qquad\leq C\,\|\mathscr{F}\|_\infty\,\|\nabla_y\mathscr{F}\|_\infty\left(\frac{1}{\epsilon}\beta(\theta^\epsilon\eta_2)+\epsilon\int_0^{\theta^\epsilon}\beta(s\eta_2)(1+s)\;\mathrm{d}s\right)\quad P_e-a.s.\end{aligned}$$

Taking $\theta^\epsilon\sim 1/\epsilon$, and using the decorrelation speed assumption (20) and the boundedness assumption (19), we are finally led to the key estimate:

$$\limsup_{\epsilon\to 0}\left\langle\sup_{|v|\in[\eta_2,1/\eta_1],y,q}\left|\langle\mathcal{L}^\epsilon(\phi^\epsilon)(y,v,q)-\mathcal{L}(\phi)(v)|\mathbb{R}^d\setminus B_{\epsilon,y,v}\rangle\right|\right\rangle=0\quad P_e-a.s..\tag{40}$$

Now things are settled enough to carry out the different steps of the proof.
Step 1.
One first introduces a vector of positive cut-off parameters $(\eta_i)_{i\in\{1,\ldots,4\}}$, $\eta_i>0$ conditioning the evolution of $t\mapsto V_t^\epsilon$, and which are constrained to vanish ($\eta_i\to 0$) eventually. The associated stopping times are similar to [23], although cut-off need to be introduced only at the macroscopic scale, which considerably simplifies the analysis. First, one looks at the first exit time of the velocity process from a compact set:

$$\tau_1^\epsilon=\inf\left\{t\in[0,T]|\,|V_t^\epsilon|\geq 1/\eta_1\right\},\tag{41}$$

and the first hitting time of a small ball at the origin:

$$\tau_2^\epsilon=\inf\left\{t\in[0,T]|\,|V_t^\epsilon|\leq \eta_2\right\}.\tag{42}$$

Then, one looks at finite variations:

$$\tau_3^\epsilon=\inf\left\{t\in[0,T]|\sup_{h\in[0,\gamma_{\eta_3}]}\left|V_t^\epsilon-V_{(t-h)\wedge 0}^\epsilon\right|\geq\eta_3\right\},\tag{43}$$

where $\gamma_{\eta_3} > 0$ is a time window associated with η_3. Finally, one needs to prevent from self-intersection by looking at the hitting time of a tubular neighborhood of the distant past trajectory of positions:

$$\tau_4^\epsilon = \inf \left\{ t \in [0,T] | \sup_{s \in [0, t-\gamma_{\eta_3}]} \left| \int_s^t V_{s'}^\epsilon \, \mathrm{d}s' \right| \leq \eta_4 \right\}. \tag{44}$$

The global stopping time is then

$$\tau^\epsilon := \tau_1^\epsilon \wedge \tau_2^\epsilon \wedge \tau_3^\epsilon \wedge \tau_4^\epsilon,$$

and the stopped process is still denoted by $V_\cdot^{\epsilon,\tau^\epsilon} = V_{\cdot \wedge \tau^\epsilon}^\epsilon$.

Step 2.
To carry out step 2 (compactness), and then step 3 (limit identification), it suffices to apply estimates (39) and (40) to Lemma 5.2 and 5.4 using some non-intersection property. The non-intersection condition is the following:

Lemma 5.6 *Assume that cut-off parameters verify:*

$$\eta_3 \leq \eta_1 \eta_2^2.$$

Then self-intersection in position at the macroscopic scale never occurs almost surely, in the sense that for any $t \in [0,T]$ and $\theta^\epsilon < \frac{\eta_4 \eta_2}{2\epsilon^3}$:

$$(Y_s^{\epsilon,\tau^\epsilon})_{s \in [0,t]} \cap B\left(Y_t^{\epsilon,\tau^\epsilon} + \theta^\epsilon V_t^{\epsilon,\tau^\epsilon}, \theta^\epsilon \left| V_t^{\epsilon,\tau^\epsilon} \right| \right) = \varnothing \quad a.s..$$

Proof. By construction in (44) (absence of self intersection in the distant past) one gets that for $0 \leq s \leq t - \gamma_{\eta_3}$ and $y \in B(Y_t^{\epsilon,\tau^\epsilon} + \theta^\epsilon V_t^{\epsilon,\tau^\epsilon}, \theta^\epsilon \left| V_t^{\epsilon,\tau^\epsilon} \right|)$:

$$\left| y - Y_s^{\epsilon,\tau^\epsilon} \right| \geq \left| Y_t^{\epsilon,\tau^\epsilon} - Y_s^{\epsilon,\tau^\epsilon} \right| - \left| y - Y_t^{\epsilon,\tau^\epsilon} \right| > \eta_4/\epsilon^3 - \frac{2\theta^\epsilon}{\eta_2} > 0,$$

and thus non-intersection is verified until time $t - \gamma_{\eta_3}$. On the other hand, we claim that for and $s \in [\gamma_{\eta_3} - t, t]$:

$$A_{s,t}^\epsilon := (Y_t^{\epsilon,\tau^\epsilon} - Y_s^{\epsilon,\tau^\epsilon}) \cdot V_t^{\epsilon,\tau^\epsilon} \geq 0.$$

Indeed,

$$A_{s,t}^\epsilon = \frac{t-s}{\epsilon^3} \left| V_t^{\epsilon,\tau^\epsilon} \right|^2 + \frac{1}{\epsilon^3} V_t^{\epsilon,\tau^\epsilon} \cdot \int_s^t V_{s'}^{\epsilon,\tau^\epsilon} - V_t^{\epsilon,\tau^\epsilon} ds',$$

and by construction in (43), one gets:

$$A_{s,t}^\epsilon \geq \frac{t-s}{\epsilon^3} \eta_2^2 - \frac{t-s}{\epsilon^3} \frac{\eta_3}{\eta_1} \geq 0,$$

which gives the result out. $\square$

Now since macroscopic non-intersection holds, one has:

$$\mathbb{E}\left(\cdot \, | V_{s \in [0,t]}^{\epsilon,\tau^\epsilon} \right) = \mathbb{E}\left(\langle \cdot \, | B_{\epsilon, Y_t^{\epsilon,\tau^\epsilon}, V_t^{\epsilon,\tau^\epsilon}} \rangle | V_{s \in [0,t]}^{\epsilon,\tau^\epsilon} \right) \quad a.s.,$$

and from Lemma 5.2 with estimates (39) and (40), one gets the tightness of ϵ-sequences of $t \mapsto V_t^{\epsilon,\tau^\epsilon}$ as soon as $\eta_3 \leq \eta_1 \eta_2^2$.

Step 3.
Extract a converging ϵ-sequence of $t \mapsto V_t^{\epsilon,\tau^\epsilon}$. The continuity of functionals involved in the definitions (41)-(44) of τ_i for $i \in \{1, ..., 4\}$ gives the convergence in distribution of the pair $((V_t^{\epsilon,\tau^\epsilon})_{t\in[0,T]}, \tau^\epsilon)$. The limit is represented by a random variable $(V_t^\tau)_{t\in[0,T]}, \tau)$ where τ is defined in the same way as (41)-(44). Using Lemma 5.4 with estimates (39) and (40) together with Lemma 5.6, we show that $t \mapsto V_t^\tau$ verifies the martingale property stopped at τ, associated with the generator $\mathcal{L}$, and thus is a Brownian motion with diffusion coefficient $\mathscr{D}$.

Step 4.
The stopping times $(\tau_4, \tau_3, \tau_2, \tau_1)$ will be removed, in this order. First consider a system of coordinates where the diffusion matrix $\mathcal{D}$ is diagonal. If $\mathcal{D}$ is singular, and since the initial condition is almost surely non null on the associated direction, the process $t \mapsto V_t^\tau$ remains in an hyperplane which evolves at constant speed, and self-intersection cannot occurs. If $\mathcal{D}$ is non-singular, using Lemma 6 of [23] (no self-intersection for hypoelliptic diffusions in dimension $d \geq 3$), one gets:

$$\lim_{\eta_4 \to 0} \mathbb{P}(\tau_4 = T) = 1.$$

The path of a Brownian motion is α-Hï¿½lder for any $0 < \alpha < 1/2$, thus denoting $\|\cdot\|_\alpha$ the Hï¿½lder norm one has by construction:

$$\mathbb{P}(\tau_3 = T) \geq \mathbb{P}\big(\|V^{\tau_3}\|_\alpha \geq \frac{\eta_3}{\gamma_{\eta_3}^\alpha}\big),$$

which tends to 1 as $\eta_3 \to 0$ with $\gamma_{\eta_3} \to 0$ sufficiently slowly. Finally, Brownian motion is non-explosive, and hits the origin with null probability for $d \geq 2$. Hence, we get

$$\lim_{\eta_1,\eta_2 \to 0} \mathbb{P}(\tau_1^0 \wedge \tau_2^0 = T) = 1.$$

Now Lemma 5.5 applies iteratively by taking $\eta_i \to 0$ with $i = 4, ..., 1$. This gives the final result. $\square$

5.3 Proof of Theorem 3.4

The proof in dimension 1 follows the same lines, yet the compensating pertubed test function is constructed directly from the inversion of the transport operator, which exists and is bounded as well as its derivative with respect to the momentum variable v, by assumption. Yet, one has to stop the process when $|V_t^\epsilon| = 0$. This stopping time cannot be removed in step 4 of the analysis, since Brownian motion in dimension 1 hits the origin with positive probability.

A L^1 framework

We designed in Section 4 a proof in the L^2 framework in order to do not obscure the arguments by tedious technical details. It is however possible to adapt the proof to different functional framework, up to slight changes in the definition of the dissipation property of the operator $\mathcal{Q}$. In particular it could be interesting to develop a proof in the L^1 functional setting. The mathematical difficulty is related to the fact that L^1 is not a reflexive Banach space and bounded sequences are relatively compact in the bigger space of bounded measures, which is not well adapted to our purposes, see Remark 4.4. Therefore, we need assumptions that guaranty weak compactness in L^1 that is to provide tightness and avoid concentration phenomena.

It turns out that considering $L \ln L$ estimates is physically sound and reaches the mathematical goal. In what follows we assume

$$\sup_{\epsilon>0} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon_{\mathrm{Init}} \left[1+|x|+|v|+\left|\ln\left(f^\epsilon_{\mathrm{Init}}\right)\right|\right] \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x \leq C < \infty. \tag{45}$$

We detail below the arguments for the the Fokker-Planck operator (10) and the Boltzmann operator (11). Having these compactness results at hand, we can readily adapt the proof of Theorem 3.2 to obtain a statement where (31) is replaced by (45) and weak convergence in L^2 is replaced by weak convergence in L^1.

A.1 The Linear Boltzmann Operator

Proposition A.1 *Consider equation* (5) *with* $\mathcal{Q}$ *given by* (11) *and* $\mathscr{F}$ *verifying* (14)–(16). *We suppose that* (45) *holds. Then, the quantities*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon \left[1+|x|+|v|+\left|\ln\left(f^\epsilon\right)\right|\right] \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x,$$
$$\frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} (f^\epsilon-\rho^\epsilon)\ \ln\left(\frac{f^\epsilon}{\rho^\epsilon}\right) \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}s$$

are bounded uniformly with respect to $\epsilon > 0$ *and* $0 < t < T < \infty$.

Proof. At first, due to (8), we have the conservation property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x = 0,$$

and the following dissipation property

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon \ln(f^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x &= \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} Q(f^\epsilon)\ln(f^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&= -\frac{\sigma}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} (f^\epsilon-\rho^\epsilon)\ln\left(\frac{f^\epsilon}{\rho^\epsilon}\right) \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&\leq 0
\end{aligned}$$

where we denote $\rho^\epsilon(t,x,v) = \int_{\mathbb{R}} f^\epsilon(t,x,v,q)\mathscr{M}(q)\ \mathrm{d}q$. This relation follows from (8) again which yields

$$\int_{\mathbb{R}} Q(f^\epsilon)\mathscr{M}(q)\ \mathrm{d}q\ \ln(\rho^\epsilon) = 0.$$

Next we need estimates controlling the tail of the distribution function f^ϵ. We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon(|x|+|v|)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&=\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\Big(v\cdot\frac{x}{|x|}+\frac{1}{\epsilon}\mathscr{F}(x/\epsilon^3,q)\cdot\frac{v}{|v|}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&=\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\ v\cdot\frac{x}{|x|}\ \mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x+\frac{1}{\epsilon}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}(f^\epsilon-\rho^\epsilon)\ \mathscr{F}(x/\epsilon^3,q)\cdot\frac{v}{|v|}\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x,
\end{aligned}$$

by using $\int_{\mathbb{R}}\mathscr{F}\mathscr{M}\ \mathrm{d}q=0$. It follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon(|x|+|v|)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&\qquad\le\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\ |v|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x+\frac{C}{\epsilon}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}|f^\epsilon-\rho^\epsilon|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x
\end{aligned}$$

holds, since $\mathscr{F}$ belongs to L^∞. We dominate the last integral by using the following elementary inequalities, which holds for any positive a,b,ϵ,ν,

$$\begin{aligned}
\frac{|a-b|}{\epsilon}&=\frac{|\sqrt{a}-\sqrt{b}|}{\epsilon}\ (\sqrt{a}+\sqrt{b})\le\frac{1}{4\nu}(\sqrt{a}+\sqrt{b})^2+\frac{\nu}{\epsilon^2}|\sqrt{a}-\sqrt{b}|^2\\
&\le\frac{1}{2\nu}(a+b)+\frac{\nu}{4\epsilon^2}(a-b)\ln(a/b)
\end{aligned}\tag{46}$$

by using the Cauchy-Schwarz inequality $(\sqrt{b}-\sqrt{a})^2=\big(\int_a^b\ \mathrm{d}y/(2\sqrt{y})\big)^2\le\frac{b-a}{4}\ln(b/a)$. It follows that

$$\begin{aligned}
&\frac{1}{\epsilon}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}|f^\epsilon-\rho^\epsilon|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&\le\frac{1}{2\nu}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}(f^\epsilon+\rho^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x+\frac{\nu}{4}\ \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}(f^\epsilon-\rho^\epsilon)\ \ln\Big(\frac{f^\epsilon}{\rho^\epsilon}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x.
\end{aligned}\tag{47}$$

Putting the pieces all together we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon(\ln(f^\epsilon)+|x|+|v|)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&\quad+\frac{\sigma-C\nu/4}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}(f^\epsilon-\rho^\epsilon)\ \ln\Big(\frac{f^\epsilon}{\rho^\epsilon}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\
&\qquad\le\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\ |v|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x+\frac{C}{2\nu}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}(f^\epsilon+\rho^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x,
\end{aligned}\tag{48}$$

where we already know that the last term is dominated by a constant, independently on ϵ, as a consequence of the mass conservation. Of course, we choose $0<\nu<4\sigma/C$. Let us set $W(x,v,q)=|x|+|v|$. Now, we use the standard trick

$$\begin{aligned}
f|\ln(f)|&=f\ln(f)-2f\ln(f)\mathbb{1}_{0\le f\le 1}\\
&\le f\ln(f)-2f\ln(f)\mathbb{1}_{0\le f\le e^{-W/4}}-2f\ln(f)\mathbb{1}_{e^{-W/4}\le f\le 1}\\
&\le f\ln(f)+\frac{W}{2}f+Ce^{-W/8},
\end{aligned}$$

where we use $(-2f\ln(f)) \leq C\sqrt{f}$ on $(0,1)$. We deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon(|\ln(f^\epsilon)| + \frac{W}{2})\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon_{\mathrm{Init}}(|\ln(f^\epsilon_{\mathrm{Init}})| + W)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x + \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\ |v|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x \\
&\qquad + t\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} e^{-W/8}\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x + \frac{C}{\nu}\ t\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon_{\mathrm{Init}}\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x
\end{aligned}$$

holds, where we used the mass conservation. It suffices to apply the Gronwall Lemma to conclude that the quantity

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon(|\ln(f^\epsilon)| + \frac{W}{2})\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x$$

is bounded uniformly with respect to $\epsilon > 0$ and $0 \leq t \leq T < \infty$. Coming back to (48), we note that

$$\frac{1}{\epsilon^2}\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} (f^\epsilon - \rho^\epsilon)\ln\Big(\frac{f^\epsilon}{\rho^\epsilon}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}s$$

is bounded uniformly with respect to $\epsilon > 0$ and $0 \leq t \leq T < \infty$ too. $\square$

Corollary A.2 *The sequence f^ϵ is weakly compact in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R})$ endowed with the measure $\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t$, and $r^\epsilon = (f^\epsilon - \rho^\epsilon\mathscr{M})/\epsilon$ is weakly compact in $L^1((0,T)\times B(0,R)\times B(0,R)\times\mathbb{R})$, endowed with the measure $\mathscr{M}(q)\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t$, for any $0 < T, R < \infty$.*

Proof. In what follows, L^1 spaces are endowed with the measure $\mathscr{M}(q)\ \mathrm{d}q$ for the variable q. By virtue of the Dunford-Pettis Theorem (see [13] Th. 4.21.2), the bounds in Proposition A.1 allow to extract a subsequence from f^ϵ which converges weakly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R})$ (and the limit actually lies in $L^\infty((0,T);L^1(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R})))$. Next, (47) proves that r^ϵ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R})$. It remains to show that no concentration can occur. To this end, we use (46) again which yields for any measurable set $A\subset(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$,

$$\begin{aligned}
\int_A |r^\epsilon|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \ \leq\ & \frac{1}{2\nu}\int_A (f^\epsilon + \rho^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \\
& +\nu\times\frac{1}{4\epsilon^2}\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} (f^\epsilon - \rho^\epsilon)\ \ln\Big(\frac{f^\epsilon}{\rho^\epsilon}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}$$

Let $\delta > 0$. The last integral is bounded uniformly with respect to ϵ, so that we can pick $\nu > 0$ verifying

$$\nu\ \sup_{\epsilon>0}\left\{\frac{1}{4\epsilon^2}\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} (f^\epsilon - \rho^\epsilon)\ \ln\Big(\frac{f^\epsilon}{\rho^\epsilon}\Big)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t\right\} \leq \delta.$$

Then, we appeal to the Dunford-Pettis Theorem and the bounds on f^ϵ which lead to

$$\sup_{\epsilon>0}\int_A (f^\epsilon + \rho^\epsilon)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \xrightarrow[|A|\to 0]{} 0.$$

Accordingly, for $|A|$ small enough we have

$$\int_A |r^\epsilon|\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}t \le 2\delta,$$

which justifies the weak compactness of r^ϵ. $\square$

A.2 The Fokker-Planck Operator

Proposition A.3 *Consider equation* (5) *with* $\mathcal{Q}$ *given by* (10) *and* $\mathscr{F}$ *verifying* (14)–(16). *We suppose that* (45) *holds. Then, the quantities*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} f^\epsilon\left[1+|x|+|v|+\left|\ln\left(f^\epsilon\right)\right|\right]\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x,$$
$$\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} \left|\partial_q\sqrt{f^\epsilon}\right|^2\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\ \mathrm{d}s$$

are bounded uniformly with respect to $\epsilon>0$ *and* $0<t<T<\infty$. *Then, the conclusions of Corollary A.2 apply to the Fokker-Planck operator as well.*

Proof. The entropy dissipation relation we obtain with the Fokker-Planck operator reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} & f^\epsilon\ln\left(f^\epsilon\right)\ \mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &+\frac{4}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}\left|\partial_q\sqrt{f^\epsilon}\right|^2\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x=0.\end{aligned}$$

Then, the proof relies on the logarithmic Sobolev inequality (see [27] Th. 8.14) which tells us that

$$\begin{aligned}0\le\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} & f^\epsilon\ln\left(\frac{f^\epsilon}{\rho^\epsilon}\right)\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &=\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}\left[\frac{f^\epsilon}{\rho^\epsilon}\ln\left(\frac{f^\epsilon}{\rho^\epsilon}\right)-\frac{f^\epsilon}{\rho^\epsilon}+1\right]\rho^\epsilon\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x\\ &\le\frac{1}{\pi}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}}\left|\partial_q\sqrt{f^\epsilon}\right|^2\mathscr{M}\ \mathrm{d}q\ \mathrm{d}v\ \mathrm{d}x.\end{aligned}$$

However, we readily check that there exists some constant $C>0$ such that for any $z\ge 0$ we have $(\sqrt{z}-1)^2\le C(z\ln(z)-z+1)$. This remark allows to adapt the previous arguments, see (47) and the proof of Corollary A.2. $\square$

B Double-Scale Convergence and L^1 Weak Compactness

This section is devoted to some technical refinements on the theory of double-scale convergence. In what follows, we consider a sequence of functions $f_n:\mathbb{R}^d\to\mathbb{R}$ verifying

$$\text{Uniform } L^1 \text{ bound:}\qquad \sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}|f_n|\ \mathrm{d}x=C_0<\infty, \tag{49}$$

$$\text{Compact support:}\qquad f_n(x)=0 \text{ for a. e. } |x|\ge R, \tag{50}$$

$$\text{Equi-integrability:} \qquad \lim_{\mathscr{L}_{\mathbb{R}^d}(A)\to 0} \left(\sup_{n\in\mathbb{N}} \int_A |f_n| \ \mathrm{d}x \right) = 0, \tag{51}$$

where here and below $\mathscr{L}_{\mathbb{R}^d}(A)$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$. The assumption (50) is not crucial in the analysis and it can be easily relaxed; here, it allows to avoid tedious difficulties due to possible loss of mass at infinity. The condition (51) can be recast as follows

$$\begin{cases} \text{There exists a (convex and non decreasing) function } G : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that} \\ \displaystyle\lim_{s\to\infty} \frac{G(s)}{s} = 0 \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} G(|f_n|) \ \mathrm{d}x = C_1 < \infty. \end{cases} \tag{52}$$

We refer to [9] (Th. 22) for details on this so-called De La Vallée Poussin criterion. Due to the Dunford-Pettis Theorem (see [13] Th. 4.21.2), when (49)-(51) are fulfiled, we already know that $\big(f_n\big)_{n\in\mathbb{N}}$ is relatively weakly compact in $L^1(\mathbb{R}^d)$.

Here, we are interested in double scale limit, defined *à la* Allaire or N'Guetseng [1, 29]. Let us denote by $\mathbb{Y}$ the unit cube in $\mathbb{R}^d$, which is endowed with the (normalized) Lebesgue measure. The symbol $\#$ is used to characterize $\mathbb{Y}-$periodicity. Given a borelian set $B \subset \mathbb{Y}$, we denote $B^\#$ its extension by periodicity to $\mathbb{R}^d$ and, for $n \in \mathbb{N}$, we will also use the notation

$$B_n^\# = \big\{x \in \mathbb{R}^d \text{ such that } nx \in B^\#\big\}$$

For $\varphi \in C^0_{c,\#}(\mathbb{R}^d \times \mathbb{Y})$, we set

$$\int_{\mathbb{R}^d\times\mathbb{Y}} \varphi(x,y) \ \ \mathrm{d}\mu_n(x,y) = \int_{\mathbb{R}^d} \varphi(x,nx) \ f_n(x) \ \mathrm{d}x$$

or in other words we consider the sequence of measures on $\mathbb{R}^d \times \mathbb{Y}^\#$

$$\mathrm{d}\mu_n(x,y) = f_n(x) \ \mathrm{d}x \otimes \delta(y = nx).$$

In view of (49), μ_n is a bounded sequence of measures and, extracting subsequences if necessary, we can suppose that it converges vaguely which means that for any $\varphi \in C^0_{\#c}\mathbb{R}^d \times \mathbb{Y}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x,nx) \ f_n(x) \ \mathrm{d}x = \int_{\mathbb{R}^d\times\mathbb{Y}} \varphi(x,y) \ \ \mathrm{d}\mu(x,y) \tag{53}$$

where μ belongs to $\mathscr{M}^1(\mathbb{R}^d \times \mathbb{Y}^\#)$. Replacing the bound (49) by a L^2 estimate, we can show that the limit is actually a function: $\mathrm{d}\mu(x,y) = F(x,y)\,dy \ \mathrm{d}x$, with $F \in L^2_\#(\mathbb{R}^d\times\mathbb{Y})$. Hence, we address the question of additional properties of the measure μ induced by the equi-integrability condition (51).

Theorem B.1 *Assume* (49)-(51)*. Then, the double scale limit is absolutely continuous with respect to the Lebesgue measure and there exists* $F \in L^1_\#(\mathbb{R}^d\times\mathbb{Y})$ *such that* $\mathrm{d}\mu(x,y) = F(x,y) \ \mathrm{d}y \ \ \mathrm{d}x$*.*

We restrict the discussion to the case where $f_n \geq 0$, and thus $\mu \geq 0$ (otherwise we apply the reasoning on the positive and negative parts...). The proof consists in proving that for any borelian set in $\mathbb{R}^d \times \mathbb{Y}$ such that $\mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(E) = 0$ and for any $\epsilon > 0$, we have $\mu(E) \leq \epsilon$. We start with elementary results of measure theory.

Lemma B.2 *Let B be a borelian set of $\mathbb{Y}$. Then, we have:*
i) For any $\Phi \in L^\infty(\mathbb{Y})$, $\sup_{x\in B_n^\#} |\Phi(nx)| \le \|\Phi\|_{L^\infty(\mathbb{Y})}$,
ii) $\mathscr{L}_{\mathbb{R}^d}(B(0,R) \cap B_n^\#) \le C(R)\ \mathscr{L}_{\mathbb{Y}}(B)$.
Finally, let A be a cube in $\mathbb{R}^d$ and B a cube in $\mathbb{Y}$. Then, we have

$$\mathscr{L}_{\mathbb{R}^d}\big(\{x \in B(0,R),\ x \in A,\ x \in B_n^\#\}\big) \le \mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(A\times B) + \frac{2^d}{n^d}\mathscr{L}_{\mathbb{Y}}(B).$$

Proof. Point i) is clear. For proving ii) we introduce a covering of $B(0,R)$ by cubes with size $1/n$ and vertices being k/n, with $k \in \mathbb{Z}^d$:

$$B(0,R) \subset \bigcup_{\ell=1}^{L_n} Q_\ell, \qquad Q_\ell = \prod_{i=1}^{d}[k_i/n, (k_i+1)/n) = (k+\mathbb{Y})/n,$$

where the number L_n of cubes necessary for the covering is of order n^d. We have

$$\mathscr{L}_{\mathbb{R}^d}(B(0,R)\cap B_n^\#) \le \sum_{\ell=1}^{L_n} \int_{Q_\ell} \mathbb{1}_{\{nx\in B^\#\}}\ \mathrm{d}x = \sum_{\ell=1}^{L_n} \int_{\mathbb{Y}} \mathbb{1}_{\{y\in B\}} \frac{\mathrm{d}y}{n^d} = C(R)\ \mathscr{L}_{\mathbb{Y}}(B).$$

The proof of the last statement follows the same argument. □

Next, we consider trial functions with separated variables.

Lemma B.3 *Let $\phi \in L^\infty(\mathbb{R}^d)$, $\mathrm{supp}(\phi) \subset B(0,R)$ and $\psi \in L^\infty_\#(\mathbb{Y})$. Then, the quantity*

$$I_n(\phi,\psi) = \int_{\mathbb{R}^d} \phi(x)\psi(nx)\ f_n(x)\ \mathrm{d}x$$

has a limit as n goes to infinity that we denote $I(\phi,\psi)$. In particular, when ϕ and ψ are continuous we get

$$I(\phi,\psi) = \int_{\mathbb{R}^d\times\mathbb{Y}} \phi(x)\psi(y)\ \mathrm{d}\mu(x,y).$$

Proof. Note that (53) already defines

$$I(\phi,\psi) = \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(x)\psi(nx) f_n(x)\ \mathrm{d}x = \int_{\mathbb{R}^d\times\mathbb{Y}} \phi(x)\psi(y)\ \mathrm{d}\mu(x,y)$$

for continuous functions $\phi \in C^0_c(B(0,R))$, $\psi \in C^0_{c,\#}(\mathbb{Y})$. By Lusin's Theorem (see [32] Th. 2. 24 & Cor. 2.24), there exist sequences of continuous functions converging $\mathscr{L}-$a. e. to ϕ and ψ respectively:

$$\begin{cases} \phi_\delta \in C^0_c(\mathbb{R}^d),\ \psi_\delta \in C^0_c(\mathbb{Y}), \\ \|\phi_\delta\|_\infty \le \|\phi\|_\infty,\ \|\psi_\delta\|_\infty \le \|\psi\|_\infty, \\ \phi_\delta(x) \xrightarrow[\delta\to 0]{} \phi(x), \qquad \psi_\delta(y) \xrightarrow[\delta\to 0]{} \psi(y) \qquad \text{for a. e. } x \in \mathbb{R}^d,\ y \in \mathbb{Y}. \end{cases}$$

Of course, we can suppose that ϕ_δ is supported in $B(0,R)$. By Egoroff's Theorem, for any $\eta > 0$, there exist measurable sets $E_\eta \subset B(0,R)$ and $F_\eta \subset \mathbb{Y}$ such that

$$\begin{cases} \phi_\delta \text{ (resp. } \psi_\delta) \text{ converge to } \phi \text{ (resp. } \psi) \text{ uniformly on } E_\eta \text{ (resp. } F_\eta), \\ \mathscr{L}_{\mathbb{R}^d}(B(0,R)\setminus E_\eta) \le \eta, \qquad \mathscr{L}_{\mathbb{Y}}(Y \setminus F_\eta) \le \eta. \end{cases}$$

Then, we write

$$
\begin{aligned}
I_n(\phi,\psi) &= \int_{\mathbb{R}^d} \big(\phi(x)-\phi_\delta(x)\big)\ \psi(nx)\ f_n(x)\ \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \phi_\delta(x)\ \big(\psi(nx)-\psi_\delta(nx)\big)\ f_n(x)\ \mathrm{d}x + \int_{\mathbb{R}^d} \phi_\delta(x)\ \psi_\delta(nx)\ f_n(x)\ \mathrm{d}x.
\end{aligned}
$$

The first term can be dominated by

$$
\|\phi-\phi_\delta\|_{L^\infty(E_\eta)}\|\psi\|_\infty \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} f_n\ \mathrm{d}x + 2\|\phi\|_\infty\|\psi\|_\infty \sup_{n\in\mathbb{N}} \int_{B(0,R)\setminus E_\eta} f_n\ \mathrm{d}x
$$

We proceed similarly with the second term, using Lemma B.2. Let $\epsilon>0$ be fixed. We first choose η small enough to guaranty

$$
2\|\phi\|_\infty\|\psi\|_\infty \sup_{n\in\mathbb{N}} \int_{B(0,R)\setminus E_\eta} f_n\ \mathrm{d}x \le \epsilon
$$

by using (51), then we pick δ small enough to obtain

$$
\|\phi-\phi_\delta\|_{L^\infty(E_\eta)}\|\psi\|_\infty \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} f_n\ \mathrm{d}x \le \epsilon.
$$

It follows that

$$
|I_n(\phi,\psi)-I_m(\phi,\psi)| \le 4\epsilon + |I_n(\phi_\delta,\psi_\delta)-I_m(\phi_\delta,\psi_\delta)|
$$

so that $\big(I_n(\phi,\psi)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence as a consequence of (53). Accordingly, $I(\phi,\psi)$ makes sense for bounded functions. □

Corollary B.4 *Let A be a compact (resp. open) set in $\mathbb{R}^d$ and let B be a compact (resp. open) set in $\mathbb{Y}$. Then, we have $\mu(A\times B) = I(\mathbb{1}_A, \mathbb{1}_B)$.*

Proof. The proof follows the same lines since characteristic functions of compact (resp. open) sets can be approached pointwise by continuous functions. □

Let $M>0$. We denote $\Lambda(M) = \sup_{s\ge M} \frac{s}{G(s)}$. Given A and B compact sets of $\mathbb{R}^d$ and $\mathbb{Y}$ respectively, we split as follows

$$
\begin{aligned}
&\int_{x\in A\cap B_n^\#} f_n(x)\ \mathrm{d}x = \int_{x\in A\cap B_n^\#} f_n(x)\ \mathbb{1}_{f_n(x)\le M}\ \mathrm{d}x + \int_{x\in A\cap B_n^\#} f_n(x)\ \mathbb{1}_{f_n(x)\ge M}\ \mathrm{d}x \\
&\le M\mathscr{L}_{\mathbb{R}^d}\big(\{x\in B(0,R), x\in A,\, nx\in B^\#\}\big) + \Lambda(M)\int_{x\in A\cap B_n^\#} G(F_n)\ \mathrm{d}x \\
&\le M\left(\mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(A\times B) + \frac{2^d}{n^d}\mathscr{L}_{\mathbb{Y}}(B)\right) + \Lambda(M)\int_{x\in A\cap B_n^\#} G(F_n)\ \mathrm{d}x.
\end{aligned}
$$

Consider a Lebesgue-negligible set $E\subset B(0,R)\times\mathbb{Y}$. For any $\eta>0$ there exists an open set $\mathscr{O}_\eta\subset\mathbb{R}^d\times\mathbb{Y}$ such that

$$
E\subset\mathscr{O}_\eta, \qquad \mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(\mathscr{O}_\eta)\le\eta.
$$

Reproducing the construction of [32], $\mathscr{O}_\eta$ can be covered by a enumerable family of boxes $A_k \times B_k$:

$$\mathscr{O}_\eta = \bigcup_{k\in\mathbb{N}} A_k \times B_k \qquad (A_k \times B_k) \cap (A_j \times B_j) = \emptyset \text{ when } n \neq j.$$

For $K \in \mathbb{N}$, we set $\mathscr{O}_\eta^K = \bigcup_{k=1}^K A_k \times B_k$, so that we get

$$0 \le \mu(E) \le \mu(\mathscr{O}_\eta) = \lim_{K\to\infty} \mu(\mathscr{O}_\eta^K).$$

However, we can write

$$\mu(\mathscr{O}_\eta^K) = \sum_{k=1}^K \mu(A_k \times B_k) \le \sum_{k=1}^K \mu(\overline{A_k} \times \overline{B_k}) \le \sum_{k=1}^K I(\mathbb{1}_{\overline{A_k}}; \mathbb{1}_{\overline{B_k}}).$$

The latter can be recast as

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sum_{k=1}^K \int_{x\in\overline{A_k}\cap(\overline{B_k})_n^\#} f_n(x)\ \mathrm{d}x \right) \\
&\le \lim_{n\to\infty} \left(\sum_{k=1}^K \left\{ M\mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(\overline{A_k}\times\overline{B_k}) + \frac{2^d}{n^d}\mathscr{L}(B_k) + \Lambda(M)\int_{x\in\overline{A_k}\cap(\overline{B_k})_n^\#} G(F_n)\ \mathrm{d}x \right\} \right) \\
&\le \lim_{n\to\infty} \left(M\mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(\mathscr{O}_\eta) + M\frac{2^d}{n^d}\sum_{k=1}^K \mathscr{L}(B_k) + \Lambda(M)\ C_1 \right).
\end{aligned}$$

Since for any $K \in \mathbb{N}$, $\sum_{k=1}^K \mathscr{L}(B_k)$ is finite we are led to

$$0 \le \mu(\mathscr{O}_\eta^K) \le M\mathscr{L}_{\mathbb{R}^d\times\mathbb{Y}}(\mathscr{O}_\eta) + \Lambda(M)C_1.$$

Let $\epsilon > 0$. We first choose M large enough to guaranty that $\Lambda(M)\ C_1 \le \epsilon/2$, and then we can pick η small enough to obtain $\mu(\mathscr{O}_\eta^K) \le \epsilon$. Since this inequality holds for any K, it finally yields

$$\mu(E) \le \mu(\mathscr{O}_\eta) \le \epsilon$$

for any positive $\epsilon > 0$ and thus $\mu(E) = 0$.

The arguments adapt readily when we take into account an additional auxiliary variable, as necessary for our purposes.

Theorem B.5 *Let $f_n : \mathbb{R}^d \times \mathbb{R}^D \to \mathbb{R}$ verifying*

Uniform L^1 bound:	$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d\times\mathbb{R}^D} \lvert f_n\rvert(x,z)\ \mathrm{d}x\ \mathrm{d}z = C_0 < \infty,$
Compact support:	$f_n(x,z) = 0$ *for a. e.* $\lvert x\rvert \ge R$, $\lvert z\rvert \ge R$,
Equi-integrability:	$\lim_{\mathscr{L}_{\mathbb{R}^d\times\mathbb{R}^D}(A)\to 0} \left(\sup_{n\in\mathbb{N}} \int_A \lvert f_n\rvert\ \mathrm{d}x\ \mathrm{d}z \right) = 0.$

Then, up to a subsequence we can assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x,nx,z)\ f_n(x,z)\ \mathrm{d}x\ \mathrm{d}z = \int_{\mathbb{R}^d\times\mathbb{Y}\times\mathbb{R}^D} \varphi(x,y,z)\ \ \mathrm{d}\mu(x,y,z)$$

holds for any $\varphi \in C^0_{c,\#}(\mathbb{R}^d \times \mathbb{Y} \times \mathbb{R}^D)$, where μ belongs to $\mathscr{M}^1(\mathbb{R}^d \times \mathbb{Y}^\# \times \mathbb{R}^D)$. Then, the double scale limit is aboslutely continuous with respect to the Lebesgue measure and there exists $F \in L^1_\#(\mathbb{R}^d \times \mathbb{Y} \times \mathbb{R}^D)$ such that $\mathrm{d}\mu(x, y, z) = F(x, y, z)\ \mathrm{d}y\ \mathrm{d}x\ \mathrm{d}z$.

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References

[1] G. Allaire. *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23, 1482–1518, 1992.

[2] B. Aguer, S. De Bièvre, in preparation, 2009.

[3] R. N. Bhattacharya *On the functional central limit theorem and the law of the iterated logarithm for Markov processes* Probability Theory and Related Fields 60 (2) 185-201 1982.

[4] L. Bruneau and S. De Bièvre. *A Hamiltonian model for linear friction in a homogeneous medium*, Commun. Math. Phys. 229, 511–542, 2002.

[5] A.-L. Dalibard. *Homogenization of a linear transport equation in a stationary ergodic setting*, Comm. PDE., 33 (5) 881–921, 2008.

[6] S. De Bièvre, P. Lafitte and P. Parris. *Normal transport at positive temperatures in classical Hamiltonian systems*, in "Adventures in Mathematical Physics: a volume in honour of J.-M. Combes", Cont. Math, 447, 57–71, 2007.

[7] S. De Bièvre, P. Lafitte and P. Parris. *Normal transport properties in a metastable stationary state for a classical particle coupled to a non-Ohmic bath*, J. Stat. Phys., 132 (5) 863–879, 2008.

[8] S. De Bièvre, P. Parris and A. Silvius. *Chaotic dynamics of a free particle interacting with a harmonic oscillator*, Physica D, 208, 96–114, 2005.

[9] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel* (Herman, 1975).

[10] D. Dolgopyat and L. Koralov. *Motion in a random force field*, Nonlinearity 22 187–211, 2009.

[11] K. Domelevo and P. Villedieu. *A hierarchy of models for turbulent dispersed two-phase flows derived from a kinetic equation for the joint particle-gas*, Comm. in Math. Sci., 5 (2) 331–353, 2007.

[12] D. Dürr, S. Goldstein, J. L. Lebowitz *Asymptotic motion of a classical particle in a random potential in two dimensions: Landau Model*, Comm. Math. Phys., 113, 209–230, 1987.

[13] R. Edwards. *Functional analysis* (Dover, 1994).

[14] S. N. Ethier and T. G. Kurtz, *Markov Processes. Characterization and Convergence* (Wiley Series in Probability and Mathematical Statistics, 1985).

[15] E. Frénod and K. Hamdache. *Homogenisation of transport kinetic equations with oscillating potentials*, Proc. Roy. Soc. Edinburgh Sect. A 126 (6) 1247–1275, 1996.

[16] F. Golse. *From kinetic to macroscopic models*, in *Kinetic Equations and Asympototic Theory*, B. Perthame and L. Desvillettes, eds. (Gauthier–Villars, Paris, France, 2000), pp. 41–126.

[17] T. Goudon and A. Mellet. *Homogenization and diffusion asymptotics of the linear Boltzmann equation*, ESAIM Control Optim. Calc. Var. 9, 371–398, 2003.

[18] T. Goudon and F. Poupaud. *Approximation by homogeneization and diffusion of kinetic equations*, Comm. PDE., 26, 537-569, 2001.

[19] T. Goudon and F. Poupaud. *Homogenization of transport equations; weak mean field approximation*, SIAM J. Math. Anal, 36 (3) 856–881, 2004.

[20] T. Goudon, P. Lafitte and M. Rousset, *Modeling and simulation of fluid-particles flows*, to appear in French-Chinese Summer Institute on Applied Mathematics, Shanghai 2008, Series in Contemporary Applied Mathematics (Editors: Ph. G. Ciarlet and Ta- Tsien Li).

[21] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes* (MIT Press, 1984).

[22] H. Kesten and G. C. Papanicolaou. *A limit theorem for turbulent diffusion*, Commun. Math. Phys. 65, 97–128, 1979.

[23] H. Kesten and G. C. Papanicolaou. *A limit Theorem for stochastic acceleration*, Comm. Math. Phys. 78, 19–63, 1980.

[24] T. Komorowski and L. Ryzhik. *The stochastic acceleration problem in two dimensions*, Israel J. Math. 155, 157–203, 2006.

[25] T. Komorowski and L. Ryzhik. *Passive tracer in a slowly decorrelating random flow with a large mean*, Nonlinearity, 20, 1215–1239, 2007.

[26] M. Krein and M. Rutman. *Linear operator leaving invariant a cone in a Banach space*, AMS Transl. 10, 199–325, 1962.

[27] E. Lieb and M. Loss. *Analysis*, Graduate Studies in Math., vol. 14, 2nd edition (AMS, Providence, 2001).

[28] G. O. Roberts and R. L. Tweedie. *Exponential convergence of Langevin distributions and their discrete approximations*, Bernoulli 2 (4) 341–363, 1996.

[29] G. Nguetseng. *A general convergence result for a functional related tothe theory of homogenization*, SIAM J. Math. Anal., 20, 608–623, 1989.

[30] F. Poupaud and A. Vasseur. *Classical and quantum transport in random media*, J. Math. Pures et Appl., 82, 711–748, 2003.

[31] H. Schaefer. *Banach lattices and positive operators* (Springer, 1974).

[32] W. Rudin. *Analyse réelle et complexe* (Masson, 1998).